Analyzing the Robustness of Impulsive Synchronization Coupled by Linear Delayed Impulses

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***Abstract*—In this paper, a class of autonomous impulsive differential systems with linear delayed impulses is considered. Sufficient conditions required for this particular class of systems with varying and constant impulse durations to be equi-attractive in the large are obtained. These conditions are then applied to impulsively synchronize two coupled chaotic systems by using delayed impulses and a robustness analysis of the model is also provided. Simulation results are given to demonstrate the analytical results.**

***Index Terms*—Impulsive synchronization, delay, equi-attractivity,**

I. INTRODUCTION

The synchronization of coupled chaotic systems has become an active research area because of its potential applications to secure communication [7], [8]. A number of interesting communication security schemes based on chaos synchronization have been proposed. In these schemes, message signals are masked or modulated (encrypted) by using chaotic signals and the resulting encrypted signals are transmitted across a public channel. An identical synchronization between the chaotic systems at the transmitter and receiver ends is required for recovering the message signal [7]. Different types of synchronization techniques (some of which are robust to parameter mismatch and channel noise) have been developed in the literature [5], [7]. Synchrony was established in some of these studies using low dimensional chaotic systems and employing first and second Lyapunov techniques.

Most recently, another synchronization technique, called impulsive synchronization (IS), has been reported in [8]. The technique allows the coupling and synchronization of two or more chaotic systems by using only small synchronizing impulses. These impulses are samples of the state variables of the drive system at discrete moments that drive the response system. When equi-attractivity in the large of the synchronization error between the drive and the response systems is achieved, the two coupled systems are said to be synchronized. This technique has been applied to a number of chaos-based secure communication schemes which exhibit good performance as far as synchronization and security are concerned [8].

In general, transmission and sampling delays in communication security schemes based on IS are inevitable. Therefore, it is very crucial to examine the robustness of IS towards these two types of delay. There have been several attempts in the literature to study the existence, uniqueness, boundedness and stability of solutions of a particular class of delayed impulsive systems [1]. In fact, the stability of linear continuous-time systems possessing delayed discrete-time controllers in networked control systems have been also analyzed [2], [6], [9]. Such studies have been based on the notions of Lyapunov-Krasovskii functionals and Lyapunov-Razumikhin functions [3]. In this paper, we investigate the stability of non-linear impulsive systems and IS in the presence of *linear delayed impulses*. By *linear delayed impulses* we mean that the mapping describing these impulsive moments (or discrete transitions) are linear in structure and dependent on delayed state variables. We derive sufficient conditions leading to synchronization when linear delayed impulses are applied. Our goal is to explore the sensitivity of IS to delayed impulses and obtain the maximum amount of delay the synchronization error could handle.

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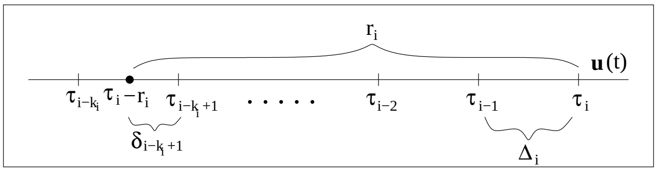

Fig. 1. Schematic figure describing the different moments and time differences.

The remainder of this paper is organized as follows. In Section 2, a general impulsive system with linear delayed impulses resembling the structure of many chaotic systems is presented and the equi-attractivity of its zero-equilibrium solution is investigated. The analysis is developed for systems with both varying and constant impulse durations. In Section 3, we present the motivation for constructing these systems and show several numerical simulations to illustrate the theory obtained in Section 4. Finally, in Section 5, we summarize our results.

II. SYSTEMS WITH LINEAR DELAYED IMPULSES

Consider the impulsive system

$$\begin{cases} \left.\begin{array}{rcll} \dot{\mathbf{x}} & = & A\mathbf{x} + \mathbf{\Phi}(t,\mathbf{x}), & t \neq \tau_i \\ \Delta\mathbf{x}(t) & = & B_i\mathbf{x}(t-r_i), & t = \tau_i \end{array}\right\} & t > t_0 \\ \mathbf{x}(t) \;=\; \boldsymbol{\phi}(t-t_0), & t_0 - r \leq t \leq t_0, \end{cases} \tag{1}$$

where A is an $n \times n$ constant matrix, $\Delta\mathbf{x}(\tau_i) = \mathbf{x}(\tau_i^+) - \mathbf{x}(\tau_i^-)$, $\mathbf{x}(\tau_i^+) = \lim_{t\to\tau_i^+}\mathbf{x}(t)$ and the moments of impulse satisfy $t_0 < \tau_1 < \tau_2 < \ldots < \tau_i < \ldots$ with $\lim_{i\to\infty}\tau_i = \infty$. The function $\boldsymbol{\phi}(t-t_0)$ is an arbitrary differentiable initial function defined over $[t_0 - r, t_0]$ and r_i are delay constants satisfying $r := \max_i(r_i) \geq 0$, $i = 1, 2, \ldots$. Let k, k_i, $i = 1, 2, \ldots$, be a set of non-negative integers chosen in such a way that $\tau_{i-k} < \tau_i - r \leq \tau_{i-k+1}$ and $\tau_{i-k_i} < \tau_i - r_i \leq \tau_{i-k_i+1}$, where $1 \leq k_i, k \leq i$ and τ_0 is defined to be some point satisfying $t_0 - r \leq \tau_0 < \tau_1$ (τ_0 does not represent a moment of impulse), as shown in Figure 1. Assume that B_i, for all $i = 1, 2, \ldots$, are $n \times n$ constant matrices satisfying $||B_i|| := \sqrt{\lambda_{\max}(B_i^TB_i)} < L_1$, for some $L_1 > 0$ ($\lambda_{\max}(B^TB)$ is the largest eigenvalue of B^TB), and that $||\mathbf{\Phi}(t,\mathbf{x})|| \leq L_2||\mathbf{x}||$, for some $L_2 > 0$. This guarantees that, for each $(t_0, \boldsymbol{\phi}) \in \mathbb{R}_+ \times C([-r,0],\mathbb{R}^n)$, there exists a local solution of (1) satisfying the initial condition $\mathbf{x}(t) = \boldsymbol{\phi}(t-t_0)$, for $t_0 - r \leq t \leq t_0$ [1]. Let $\mathbf{x}(t) := \mathbf{x}(t, t_0, \boldsymbol{\phi})$ be any solution of (1) satisfying $\mathbf{x}(t) = \boldsymbol{\phi}(t-t_0)$, for $t_0 - r \leq t \leq t_0$, and $\mathbf{x}(t)$ be left continuous at each $\tau_i > t_0$ in the interval of existence, i.e., $\mathbf{x}(\tau_i^-) = \mathbf{x}(\tau_i)$, $i = 1, 2, \ldots$. Using the above set up, we define the impulse interval Δ_i by $\Delta_i := \tau_i - \tau_{i-1}$, and the quantities δ_{i-k_i+1}, by $\delta_{i-k_i+1} := r_i - (\tau_i - \tau_{i-k_i+1})$, where $0 \leq \delta_{i-k_i+1} < \Delta_{i-k_i+1}$, $i = 1, 2, \ldots$, as shown in Figure 1. The latter expressions and the following classes of functions and definitions, are very necessary to state the main results of this paper. Let $S^c(M) := \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| \geq M\}$, $S^c(M)^0 := \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| > M\}$ and $\nu_0(M) := \{V : \mathbb{R}_+ \times S^c(M) \to \mathbb{R}_+ : V(t,\mathbf{x}) \in C((\tau_i, \tau_{i+1}] \times S^c(M))$, locally Lipschitz in $\mathbf{x}$ and $V(\tau_i^+, \mathbf{x})$ exists for $i = 1, 2, \ldots\}$, where $M \geq 0$.

Definition 1: [3] Given the delay constant r, we equip the linear space $C([-r,0],\mathbb{R}^n)$ with the norm $||\cdot||_r$ defined by $||\boldsymbol{\phi}||_r = \sup_{-r\leq s\leq 0}||\boldsymbol{\phi}(s)||$.

Definition 2: Let $M \geq 0$ and $V \in \nu_0(M)$. Define the upper right derivative of $V(t,\mathbf{x})$ with respect to the continuous portion of (1) by

$$D^{+}V(t,\mathbf{x}) := \lim_{\delta\to 0^{+}} \sup \frac{1}{\delta}\left[V(t+\delta,\mathbf{x}+\delta\mathbf{f}(t,\mathbf{x})) - V(t,\mathbf{x})\right],$$

for $(t,\mathbf{x}) \in \mathbb{R}_{+} \times S^{c}(M)^{0}$ and $t \neq \tau_i$, where $\mathbf{f}(t,\mathbf{x}) := A\mathbf{x} + \mathbf{\Phi}(t,\mathbf{x})$.

Definition 3: Solutions of (1) are said to be

(S1) equi-attractive in the large if for each $\epsilon > 0$, $\alpha > 0$ and $t_0 > 0$, there exists a number $T := T(t_0,\epsilon,\alpha) > 0$ such that $||\boldsymbol{\phi}||_r < \alpha$ implies $||\mathbf{x}(t)|| < \epsilon$, for $t \geq t_0 + T$;

(S2) uniformly equi-attractive in the large if T in (S1) is independent of t_0.

From Definition 3, we conclude that solution trajectories of (1) satisfy $\lim_{t\to\infty}\mathbf{x}(t) = \mathbf{0}$ no matter how large the $||\boldsymbol{\phi}||_r$. By applying the Mean Value Theorem and (1), we obtain

$$\mathbf{x}(\tau_i - r_i) = \mathbf{x}(\tau_{i-k_i+1}) - \delta_{i-k_i+1}A\mathbf{x}(t) - \delta_{i-k_i+1}\mathbf{\Phi}(t,\mathbf{x}(t)),$$

for $0 \leq \delta_{i-k_i+1} < \Delta_{i-k_i+1}$ and for some $t \in (\tau_{i-k_i+1} - \delta_{i-k_i+1}, \tau_{i-k_i+1}) \subset (\tau_{i-k_i}, \tau_{i-k_i+1})$, $i = 1,2,\ldots$. Thus

$$\begin{aligned} ||\mathbf{x}(\tau_i) - \mathbf{x}(\tau_i - r_i)|| \quad \leq \quad & ||\mathbf{x}(\tau_i) - \mathbf{x}(\tau_{i-k_i+1})|| + \\ & \delta_{i-k_i+1}(||A|| + L_2)||\mathbf{x}(t)||, \end{aligned} \tag{2}$$

for some $t \in (\tau_{i-k_i+1} - \delta_{i-k_i+1}, \tau_{i-k_i+1})$ and for $i = 1,2,\ldots$. Furthermore, by applying the Fundamental Theorem of Calculus on (1) and using Schwarz-Hölder inequality, we have

$$||\mathbf{x}(\tau_i) - \mathbf{x}(\tau_{i-1}^{+})||^2 \leq \Delta_i(||A|| + L_2)^2 \int_{\tau_{i-1}^{+}}^{\tau_i} ||\mathbf{x}(t)||^2 dt, \tag{3}$$

for $i = 1,2,\ldots$. The importance of (2) and (3) will eventually become evident in the proof of the next theorem.

Notice that (1) has varying impulse durations and varying delay terms corresponding to each impulse. Therefore, equipped with the above definitions and results, we shall derive in Theorem 1 the sufficient conditions needed to establish the uniform equi-attractivity property for this case. Then, in Corollary 1, we shall derive similar conditions for (1) when all the impulses are equidistant (i.e., $\Delta_i = \Delta$, $i = 1,2,\ldots$) and all the delay terms are equal (i.e., $r_i = r$, $i = 1,2,\ldots$).

Theorem 1: Let $2\widetilde{\lambda}$ be the largest eigenvalue of $A^T + A$, $\Delta_i \leq \Delta$ for some $\Delta > 0$,

$$\mathcal{K}_i := e^{a\Delta_i}||I + B_i||, \tag{4}$$

$$\mathcal{F}_i^{(j)} := \Delta_j e^{a\Delta_j}||B_i||(||A|| + L_2), \quad i - k_i + 2 \leq j \leq i, \quad k_i > 1, \tag{5}$$

$$\mathcal{E}_i^{(j)} := e^{a\delta_{j-k_j+1}}||B_i||||B_j||, \quad i - k_i + 1 \leq j \leq i-1, \quad k_i > 1, \tag{6}$$

$$\mathcal{L}_i := \delta_{i-k_i+1}e^{a\Delta_{i-k_i+1}}||B_i||(||A|| + L_2), \tag{7}$$

$i = 1,2,\ldots$, where $a := \widetilde{\lambda} + L_2$ and I is the identity matrix with the appropriate dimensions. Let J_i be the $(i - p_i) \times (i - p_i)$ matrix (where $p_i = \max(0, q_i)$ and $q_i = \min(i - k_i, i - 1 - k_{i-1}, i - 2 - k_{i-2}, \ldots, i - k_i + 1 - k_{i-k_i+1})$ given by

$$\left\{ \begin{array}{ll} J_i & := (\mathcal{K}_i + \mathcal{L}_i), \text{ if } i = 1 \text{ or } k_i = 1 \text{ (i.e., } p_i = i - k_i\text{), or} \\ J_i & := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_i^{(1)} & \alpha_i^{(2)} & \alpha_i^{(3)} & \cdots & \alpha_i^{(i-p_i)} \end{pmatrix}, \text{ if } k_i > 1, \end{array} \right. \tag{8}$$

where

$$\begin{aligned} \alpha_i^{(j)} &:= \sum_{S_{i,j}} \mathcal{E}_i^{(S_{i,j})} \ (1 \leq j \leq i - k_i - p_i), \\ \alpha_i^{(i-k_i-p_i+1)} &:= \mathcal{L}_i + \sum_{S_{i,i-k_i-p_i+1}} \mathcal{E}_i^{(S_{i,i-k_i-p_i+1})}, \\ \alpha_i^{(j)} &:= \mathcal{F}_i^{(j+p_i)} + \sum_{S_{i,j}} \mathcal{E}_i^{(S_{i,j})} \ (i - k_i - p_i + 2 \leq j \leq i - p_i - 1), \\ \alpha_i^{(i-p_i)} &:= \mathcal{K}_i + \mathcal{F}_i^{(i)}, \end{aligned}$$

and the set of integers $S_{i,j}$ (which could be empty) depends on the impulse i and the index j which both relate to the delay terms r_i, for all i such that $k_i > 1$. Then (1) is uniformly equi-attractive in the large if every eigenvalue, $\lambda^{(i)}$, of J_i satisfies $|\lambda^{(i)}| \leq \gamma$, where $0 \leq \gamma < 1$, $i = 1,2,\ldots$ (i.e., the matrix J_i can define a contraction linear mapping).

Proof: We shall use the Euclidean norm as a Lyapunov function to prove this theorem. According to Definition 2, we have

$$D^{+}||\mathbf{x}|| = \frac{1}{2}(\mathbf{x}^T\mathbf{x})^{-1/2}\Big[\dot{\mathbf{x}}^T\mathbf{x} + \mathbf{x}^T\dot{\mathbf{x}}\Big] \leq a||\mathbf{x}||. \tag{9}$$

Thus, by (9) and for every $t \in (\tau_{i-1}, \tau_i]$, $i = 1,2,\ldots$, we have

$$||\mathbf{x}(t)|| \quad \leq \quad e^{a(t-\tau_{i-1})}||\mathbf{x}(\tau_{i-1}^{+})|| \leq e^{a\Delta_i}||\mathbf{x}(\tau_{i-1}^{+})||. \tag{10}$$

Let us first consider the case when $i = 1$. By (2) and (10), and for some $t^* \in (\tau_1 - r_1, \tau_1)$, we obtain

$$\begin{aligned} ||\mathbf{x}(\tau_1^{+})|| \quad \leq \quad & e^{a\Delta_1}||I + B_1||||\mathbf{x}(\tau_0^{+})|| + \\ & r_1 e^{a\Delta_1}||B_1||(||A|| + L_2)||\mathbf{x}(\tau_0^{+})||, \end{aligned}$$

since $\tau_0 < \tau_1 - r_1$ (recall that $\mathbf{x}(\tau_0^{+}) = \mathbf{x}(\tau_0)$). Thus

$$||\mathbf{x}(\tau_1^{+})|| \leq (\mathcal{K}_1 + \mathcal{L}_1)||\mathbf{x}(\tau_0^{+})||. \tag{11}$$

Similarly, if $k_i = 1$, for some $i = 2,3,\ldots$, we may also conclude that

$$||\mathbf{x}(\tau_i^{+})|| \leq (\mathcal{K}_i + \mathcal{L}_i)||\mathbf{x}(\tau_{i-1}^{+})||. \tag{12}$$

However, if $k_i > 1$, for some $i = 2,3,\ldots$, we have to apply (1), Schwarz inequality, (2) and (10) as follows. For $t \in (\tau_{i-k_i+1} - \delta_{i-k_i+1}, \tau_{i-k_i+1})$, where $0 \leq \delta_{i-k_i+1} < \Delta_{i-k_i+1}$, we have

$$\begin{aligned} ||\mathbf{x}(\tau_i^{+})|| \quad \leq \quad & e^{a\Delta_i}||I + B_i||||\mathbf{x}(\tau_{i-1}^{+})|| + \\ & ||B_i||||\mathbf{x}(\tau_i) - \mathbf{x}(\tau_{i-k_i+1})|| + \\ & \delta_{i-k_i+1}e^{a\Delta_{i-k_i+1}}||B_i||(||A|| + L_2)||\mathbf{x}(\tau_{i-k_i}^{+})||. \end{aligned} \tag{13}$$

Moreover, if $k_i > 1$, for some $i = 2,3,\ldots$, we have

$$\begin{aligned} ||\mathbf{x}(\tau_i) - \mathbf{x}(\tau_{i-k_i+1})|| \quad \leq \quad & \sum_{j=i-k_i+2}^{i} ||\mathbf{x}(\tau_j) - \mathbf{x}(\tau_{j-1}^{+})|| + \\ & \sum_{j=i-k_i+1}^{i-1} ||B_j||||\mathbf{x}(\tau_j - r_j)||. \end{aligned}$$

Thus, by (3) and (10), we get

$$\begin{aligned} ||\mathbf{x}(\tau_i) - \mathbf{x}(\tau_{i-k_i+1})|| \leq \quad & \sum_{j=i-k_i+2}^{i} \Delta_j e^{a\Delta_j}(||A|| + L_2)||\mathbf{x}(\tau_{j-1}^{+})|| \\ & + \sum_{j=i-k_i+1}^{i-1} e^{a\delta_{j-k_j+1}}||B_j||||\mathbf{x}(\tau_{j-k_j}^{+})||. \end{aligned}$$

Hence, by the latter inequality, (13), (4), (5), (6) and (7), we have

$$\begin{aligned} ||\mathbf{x}(\tau_i^{+})|| \leq \quad & \mathcal{K}_i||\mathbf{x}(\tau_{i-1}^{+})|| + \sum_{j=i-k_i+2}^{i} \mathcal{F}_i^{(j)}||\mathbf{x}(\tau_{j-1}^{+})|| + \\ & \sum_{j=i-k_i+1}^{i-1} \mathcal{E}_i^{(j)}||\mathbf{x}(\tau_{j-k_j}^{+})|| + \mathcal{L}_i||\mathbf{x}(\tau_{i-k_i}^{+})||, \end{aligned} \tag{14}$$

for all i such that $k_i > 1$. Since $q_i = \min(i-k_i, i-1-k_{i-1}, \dots, i-k_i+1-k_{i-k_i+1})$ and $p_i = \max(0, q_i)$, we may define the quantities $v_1(p_i) := ||\mathbf{x}(\tau^+_{p_i})||$, $v_2(p_i) := ||\mathbf{x}(\tau^+_{p_i+1})||$, $\dots$, $v_{i-p_i}(p_i) := ||\mathbf{x}(\tau^+_{i-1})||$. It follows that

$$v_j(p_i+1) = ||\mathbf{x}(\tau^+_{p_i+j})|| = v_{j+1}(p_i), \tag{15}$$

where $1 \le j \le i - p_i$. By considering now (14) and (15), we obtain

$$\begin{aligned} v_{i-p_i}(p_i+1) \le \quad & \mathcal{K}_i v_{i-p_i}(p_i) + \sum_{j=i-k_i+2}^{i} \mathcal{F}_i^{(j)} v_{j-p_i}(p_i) + \\ & \sum_{i-k_i+1}^{i-1} \mathcal{E}_i^{(j)} v_{j+1-k_j-p_i}(p_i) + \mathcal{L}_i v_{i-k_i-p_i+1}(p_i). \end{aligned}$$

Let $\mathbf{v}(p_i) := (v_1(p_i), v_2(p_i), \dots, v_{i-p_i}(p_i))^T$. Then, by (8), the system of difference equations obtained above together with (11) and (12) can be expressed as

$$\mathbf{v}(p_i+1) \le J_i \mathbf{v}(p_i), \tag{16}$$

for all $i = 1, 2, \dots$, where the inequality holds componentwise. Thus if each eigenvalue, $\lambda^{(i)}$, of J_i, $i = 1, 2, \dots$, satisfies $|\lambda^{(i)}| \le \gamma$, $0 \le \gamma < 1$, then, by (16), it follows that $\lim_{s\to\infty} \mathbf{v}(p_i + s) = \mathbf{0}$, since $k_i < k$. On the other hand, $v_{i-p_i}(p_i+s) = ||\mathbf{x}(\tau^+_{i+s-1})||$, for $s = 1, 2, \dots$. Therefore if we let $\ell = i + s - 1$, we can conclude that

$$\lim_{\ell\to\infty} ||\mathbf{x}(\tau^+_\ell)|| = \lim_{s\to\infty} ||\mathbf{x}(\tau^+_{i+s-1})|| = \lim_{s\to\infty} v_{i-p_i}(p_i+s) = 0.$$

Moreover, from (10), we can further conclude that for every $t \in (\tau_{i-1}, \tau_i]$ and $i = 1, 2, \dots$, we have $||\mathbf{x}(t)|| \le e^{a\Delta_i} ||\mathbf{x}(\tau^+_{i-1})|| \le e^{a\Delta} ||\mathbf{x}(\tau^+_{i-1})|| \to 0$, as $i \to \infty$. i.e., $||\mathbf{x}(t)|| \to 0$, as $t \to \infty$. Thus solutions to (1) are uniformly equi-attractive in the large. □

The complexity of the terms given in Theorem 1 reduces drastically when considering an impulsive system with equidistant impulses and a fixed delay term at every impulse. i.e.,

$$\left\{ \begin{array}{rcll} \dot{\mathbf{x}} & = & A\mathbf{x} + \mathbf{\Phi}(t, \mathbf{x}), & t \ne \tau_i \\ \Delta\mathbf{x}(t) & = & B_i \mathbf{x}(t-r), & t = \tau_i \\ \mathbf{x}(t) & = & \boldsymbol{\phi}(t - t_0), & \end{array} \right. \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \quad t > t_0 \\ \quad t_0 - r \le t \le t_0, \end{array} \tag{17}$$

where $\tau_{i+1} - \tau_i = \Delta$ and $r_i = r$, for all $i = 1, 2, \dots$. Thus $k_i = k$ and $\delta_i = \delta$. By using the above set up, we have the following corollary.

Corollary 1: Let $2\widetilde{\lambda}$ be the largest eigenvalue of $A^T + A$, $\Delta_i = \Delta$, for some $\Delta > 0$,

$$\mathcal{K}_i := e^{a\Delta} ||I + B_i||, \tag{18}$$

$$\mathcal{F}_i^{(j)} := \Delta e^{a\Delta} ||B_i|| (||A|| + L_2), \ \ i-k+2 \le j \le i, \ \ k > 1, \tag{19}$$

$$\mathcal{E}_i^{(j)} := e^{a\delta} ||B_i|| ||B_j||, \ \ i-k+1 \le j \le i-1, \ \ k > 1, \tag{20}$$

$$\mathcal{L}_i := \delta e^{a\Delta} ||B_i|| (||A|| + L_2), \tag{21}$$

$i = 1, 2, \dots$, where $a := \widetilde{\lambda} + L_2$ and I is the identity matrix with the appropriate dimensions. Let J_i be the $(2k-1) \times (2k-1)$ matrix given by

$$\left\{ \begin{array}{lll} J_i & := (\mathcal{K}_i + \mathcal{L}_i), \text{ if } k = 1, \text{ or} \\ J_i & := \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ \alpha_i^{(1)} & \alpha_i^{(2)} & \alpha_i^{(3)} & \cdots & \alpha_i^{(2k-1)} \end{pmatrix}, \text{ if } k > 1, \end{array} \right. \tag{22}$$

where

$$\begin{aligned} &\alpha_i^{(j)} := \mathcal{E}_i^{(i-k+j)}, \ (1 \le j \le k-1), \\ &\alpha_i^{(k)} := \mathcal{L}_i, \\ &\alpha_i^{(j)} := \mathcal{F}_i^{(i+j-2k+1)}, \ (k+1 \le j \le 2k-2), \\ &\alpha_i^{(2k-1)} := \mathcal{K}_i + \mathcal{F}_i^{(i)}. \end{aligned}$$

Then (17) is uniformly equi-attractive in the large if every eigenvalue, $\lambda^{(i)}$, of J_i satisfies $|\lambda^{(i)}| \le \gamma$, where $0 \le \gamma < 1$, $i = 1, 2, \dots$.

These results are similar to already existing theories on the stability of linear continuous-time systems with discrete-time controllers possessing delays [2], [6], [9]. Such studies were based on the notions of Lyapunov-Krasovskii functionals and Lyapunov-Razumikhin functions [3]. The models encountered in these theories were specific type of linear impulsive systems used in Networked Control Systems (NCSs), whose stability is determined by solving a set of Linear Matrix Inequalities (LMIs), using existing toolboxes, or by applying Schur criterion. In this current study, however, non-linear impulsive models have been considered and trajectory-based techniques have been employed to investigate stability of these models under different delay conditions that may well-exceed impulse durations. Lipschitz-type condition, an inherent property of the chaotic systems used in these models (see Section 3), have been imposed on these non-linearities to make such arguments feasible. This condition makes the non-linear impulsive models described by (1) behave locally like a linear system in the vicinity of the equilibrium point $\mathbf{0}$. In other words, the LMIs obtained in [9], for example, could be applied locally to (1) to derive stability conditions in the neighborhood of $\mathbf{0}$ in such a way that the NCS described in [9] becomes

$$Z(t) = \overline{A} Z(t) + \overline{B} Z(t - \tau(t)) + \overline{\mathbf{\Phi}}(t, Z(t)),$$

where $Z(t) = (\mathbf{x}(t), \mathbf{e}(t))^T$, $\mathbf{e}(t) = \mathbf{x}(t) - \widehat{\mathbf{x}}(t)$, $\overline{\mathbf{\Phi}}(t, Z(t)) = (\mathbf{\Phi}(t, \mathbf{x}(t)), \mathbf{\Phi}(t, \mathbf{x}(t)))^T$ and $\tau(t)$ is the delay due to data-packet dropout. Here $\mathbf{x}$ is the plant given by the ODE in (1) and $\widehat{\mathbf{x}}$ is the impulsive controller.

With Theorem 1 and Corollary 1, we have the following five useful remarks.

Remark 1: If $k_i = 1$ (or $k = 1$), for some $i = 1, 2, \dots$, then the matrix J_i corresponding to this i^{th} impulse will be the 1×1 matrix given by $J_i = (\mathcal{K}_i + \mathcal{L}_i)$. However, if $k_i > 1$ (or $k > 1$), for some $i = 2, 3, \dots$, then the eigenvalues of the matrix J_i defined by (8) (and also by (22)) are the roots of the characteristic equation given by

$$\det(\lambda^{(i)} I - J_i) = \left(\lambda^{(i)}\right)^{i-p_i} - \sum_{j=1}^{i-p_i} \left(\lambda^{(i)}\right)^{i-p_i-j} \alpha_i^{(i-p_i+1-j)} = 0. \tag{23}$$

In fact, the matrix J_i is called the companion matrix of the polynomial given by the left hand side of (23). We may find the roots of this characteristic equation numerically and choose the matrices B_i in such a way that all the eigenvalues, $\lambda^{(i)}$, lie inside a circle of radius γ. The choice we make for the values of k_i, $i = 2, 3, \dots$, is determined by trial and error. It is important to point out that, although Theorem 1 implies that, in theory (i.e., the general case), we need to check the eigenvalues of an infinite but countable number of matrices J_i, in practice, we do not need to do so. Variation in the delay terms r_i appearing in the impulses is not significant, as it is an outcome of our own design. Therefore we may consider worst case scenario by taking the largest delay recorded by the system and apply equidistant impulses with constant magnitude $B = B_i$, $i = 1, 2, \dots$, as described by Corollary 1.

Remark 2: From (4), (5), (6) and (7) (similarly from (18), (19), (20) and (21)), we see clearly that as the delay terms r_i, $i = 1, 2, \dots$, increase, the magnitude of the impulses, $||B_i||$, must be chosen small

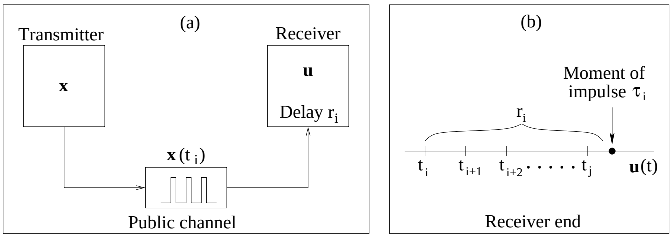

Fig. 2. (a) In cryptosystems based on IS, the impulses are transmitted across public channels but arrive at the receiver end with a time delay given by r_i. (b) Schematic description of the exact moment of impulse τ_i.

enough in order to maintain the equi-attractivity property of (1) (and (17)) provided that Δ_i are kept unchanged. This is due to the fact that for small $||B_i||$, $i = 1, 2, \dots$, $\mathcal{F}_i^{(j)}$, $\mathcal{E}_i^{(j)}$ and $\mathcal{L}_i$ will become relatively small, whereas $\mathcal{K}_i$ will become the dominant term. In other words, by choosing small $||B_i||$, we can reduce the influence of all the impulses preceding the i^{th} impulse and increase the influence of the i^{th} impulse itself. However, there is a minimum threshold value β_i for $||B_i||$ below of which the desired equi-attractivity property will not be achieved (while keeping Δ_i unchanged), since $\mathcal{K}_i$ will become large and consequently making $|\lambda^{(i)}| > 1$. This means that for large enough r_i and a given fixed value for each Δ_i, $i = 1, 2, \dots$, one may not be able to find a suitable matrix B_i which can drive the solutions of (1) (and (17)) to zero. This phenomenon will be illustrated in the next section by an example employing two Chua's oscillators.

Remark 3: The conditions of Theorem 1 and Corollary 1 are sufficient but not necessary conditions. Therefore, in the process of searching for impulse durations Δ_i and matrices B_i, $i = 1, 2, \dots$, which guarantee equi-attractiveness, we may apply Theorem 1 (or Corollary 1) to find such B_i. However, one may still be able to find other matrices B_i that fail the conditions of Theorem 1 (or Corollary 1) but remain capable of generating equi-attractive solutions.

III. Simulation Results

In cryptosystems based on IS, only samples of the drive chaotic system at the discrete moments t_i are transmitted across a public channel to the receiver end (see Figure 2(a)). This set of discrete values $\{\mathbf{x}(t_i)\}$, $i = 1, 2, \dots$, are used to linearly and impulsively drive the response chaotic system and to synchronize it with the drive system. In other words, the state variables of the response system are subjected to 'linear-type' jumps at these moments in order to make them mimic the behaviour of the driving chaotic system. According to this design (see Figure 2(b)), there are two types of delay involved in the model: (a) transmission delay due to transmitting the impulses through a public channel; and (b) sampling delay defined as being the time it takes for the system $\mathbf{u}(t)$ to produce (or sample) its values at each moment t_i and to formulate the difference $\Delta\mathbf{e}(t_i) = \Delta\mathbf{x}(t_i) - \Delta\mathbf{u}(t_i)$, $i = 1, 2, \dots$. Due to the fact that these two kinds of delay co-exist, it is not feasible to apply the impulses at the exact moments t_i to achieve synchronization. Rather, there will be a delay term, r_i, at each impulse i, $i = 1, 2, \dots$, representing the maximum of the two types of delay involved in the system, in general, and in the impulses, in particular. Therefore one may conclude that the impulses are, in fact, applied at the moments $\tau_i := t_i + r_i$, for some $r_i \geq 0$, $i = 1, 2, \dots$.

With the above set up, the general expression of the chaotic systems at the transmitter and receiver ends are given by

$$\text{Transmitter}: \dot{\mathbf{x}} = A\mathbf{x} + \mathbf{\Omega}(\mathbf{x}) \tag{24}$$

and

$$\text{Receiver}: \left\{ \begin{array}{rcll} \dot{\mathbf{u}} & = & A\mathbf{u} + \mathbf{\Omega}(\mathbf{u}), & t \neq \tau_i \\ \Delta\mathbf{u}(t) & = & -B_i\mathbf{e}(t - r_i), & t = \tau_i,\ i = 1, 2, \dots, \end{array} \right. \tag{25}$$

where $\mathbf{\Omega}(\mathbf{x})$ is a continuous non-linear mapping from $\mathbb{R}^n \to \mathbb{R}^n$ and $\mathbf{e} = \mathbf{x} - \mathbf{u}$. Observe that the general expressions in (24) and (25) include many well-known chaotic systems such as the Lorenz chaotic attractor, the Chua's oscillator and the Rössler system. Moreover, notice the presence of the delay term in the impulses and their linear nature shown in the second part of (25). By using (24) and (25), it is easy to see that the synchronization error $\mathbf{e}$ is given by

$$\left\{ \begin{array}{rcll} \dot{\mathbf{e}} & = & A\mathbf{e} + \mathbf{\Phi}(\mathbf{x}, \mathbf{u}), & t \neq \tau_i \\ \Delta\mathbf{e}(t) & = & B_i\mathbf{e}(t - r_i), & t = \tau_i,\ i = 1, 2, \dots, \end{array} \right. \tag{26}$$

where $\mathbf{\Phi}(\mathbf{x}, \mathbf{u}) = \mathbf{\Omega}(\mathbf{x}) - \mathbf{\Omega}(\mathbf{u})$. It should be mentioned here that if any of the chaotic systems listed above is used (or any other chaotic system for that matter), then the mapping $\mathbf{\Phi}$ will satisfy the following property: $||\mathbf{\Phi}(\mathbf{x}, \mathbf{u})|| \leq L_2||\mathbf{e}||$, for some $L_2 > 0$. The latter property would follow from the Mean Value Theorem and the fact that chaotic systems (including hyperchaotic and spatiotemporal chaotic systems) are bounded in the state space. Notice first that (26) resembles (1) in structure. Therefore one could apply the theory developed in Section 2 on (26). Second, (24) and (25) will impulsively synchronize whenever the error $\mathbf{e} = \mathbf{x} - \mathbf{u}$, given by (26), is equi-attractive in the large, or satisfies $\lim_{t\to\infty} \mathbf{e}(t) = \mathbf{0}$. We aim to examine the influence of delay in reaching this goal by applying the theory developed in the previous section, particularly Corollary 1.

In the following set of numerical examples, we employ a fourth order Runge-Kutta method with step size 10^{-5} and choose Chua's oscillator as the chaotic attractor. We also make the physically reasonable assumption that $r_i < r$ for some $r > 0$ and for all $i = 1, 2, \dots$. The Chua's oscillator used in these examples and simulations is given by

$$\dot{x} = 15(y - x - f(x)), \quad \dot{y} = x - y + z, \quad \dot{z} = -20y - 0.5z,$$

where $f(x) = (1125/7)x + (675/x)(|x + 1| - |x - 1|)$ and $\boldsymbol{\phi}(t - t_0) = \boldsymbol{\phi}(t) = (-2.436, 0.345, 1.639)^T$. In this case, the parameters of the system are $2\widetilde{\lambda} \approx 20.1622$, $||A|| \approx 26.9791$ and $||\mathbf{\Phi}(\mathbf{x}, \mathbf{u})|| \leq L_2||\mathbf{e}||$, where $L_2 = 1800/7$ (see [8]), i.e., $a \approx 267.2239$. Suppose that the delay term $r_i = r$ is constant (i.e., $k_i = k$), $\Delta_i = \Delta = 0.002$ is also a constant and that $B_i := B = -I$, for all $i = 1, 2, \dots$. Then, from Corollary 1 and by (23), $\mathcal{K}_i = 0$, $\mathcal{F}_i^{(j)}$ $\mathcal{E}_i^{(j)}$ are not applicable (because $k = 1$), $\mathcal{L}_i = \mathcal{L} = re^{a\Delta}(||A|| + L_2)$ and $J_i = J = (\mathcal{L})$, for all $i = 1, 2, \dots$. In this case, the error $\mathbf{e}$ is uniformly equi-attractive in the large (or the two chaotic systems $\mathbf{x}$ and $\mathbf{u}$ are impulsively synchronized) if $r_i \leq r_{\max} \approx e^{-a\Delta}\gamma/(||A|| + L_2) = (2.0625 \times 10^{-3})\gamma$, which predicts the maximum amount of delay permissible. The parameter γ can be chosen to be very close to one but less than one. In fact, by letting $\gamma := 0.999$, we obtain$r_i \leq r_{\max} \approx 2.0604 \times 10^{-3}$.

As mentioned earlier, if the delay terms r_i are less or equal to the above value, solutions will always be uniformly equi-attractive in the large. The solutions might still remain uniformly equi-attractive in the large for delay terms r_i slightly bigger than $r_{\max}$, as indicated in Remark 3. But as r gets significantly bigger, the equi-attractivity property will be lost at one point. This is shown clearly in Figure 3. The solid curve represents the first component e_1 of the error $\mathbf{e}$, whereas the dashed curve represents the second component e_2 and the dashed-dotted curve represents the third component e_3. We see that in (a), for $B = -I$ and $r = 0.0011$, solutions quickly converge to zero in 0.031 seconds, as we have predicted. However, in (b), we see that, for $r = 0.0021 > r_{\max}$, uniform equi-attractivity in the large is still achieved and solutions approach zero in 3.1 seconds.

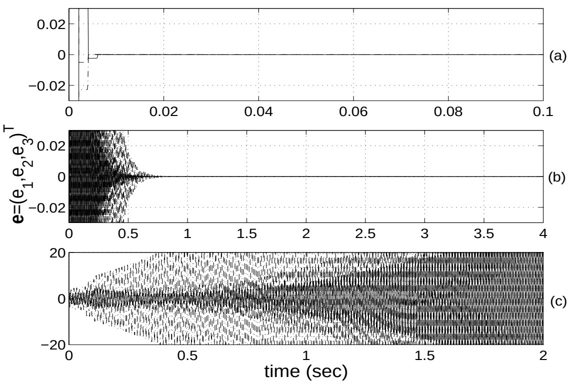

Fig. 3. IS of two Chua's oscillators with different delay values (a) $r = 0.0011$. (b) $r = 0.0021$. (c) $r = 0.003$.

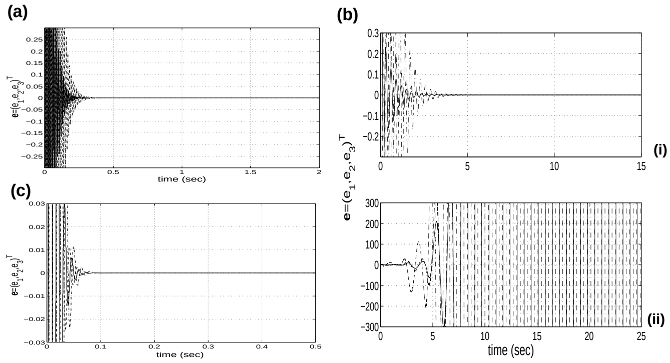

Fig. 4. (a) IS of two Chua's oscillators with $B = -0.7I$ and $r = 0.003$. (b) IS of two Chua's oscillators: (i) $B = -0.03I$ and $r = 0.07$; (ii) $B = -0.002I$ and $r = 2$. (c) IS of two Chua's oscillators with $B = -I$, $r = 0.003$ and $\Delta = 0.001$.

Finally, increasing r significantly beyond this value will make the synchronization error oscillate around zero but never approach zero, as shown in (c), where $r = 0.003$. On the other hand, if we decrease $||B_i||$ to a value less than 1, say $B = -0.7I$, in the latter case when $r = 0.003$, the equi-attractivity property is once again reached, as predicted by Corollary 1 and as shown in Figure 4(a).

When the delay term r increases to a relatively large value and Δ is kept fixed, we have to choose the matrices $B_i = B$, $i = 1, 2, \ldots$, so that $||B||$ must be small enough. For example, in Figure 4(b), panel (i), we see that if the delay term r is taken to be 0.07, then we must choose $||B|| = 0.03$ in order to impulsively synchronize the two Chua's oscillators. This is due to the fact that we need to keep the influence of the impulses preceding the i^{th} impulse negligible, as indicated in Remark 2. Furthermore, increasing the delay term r to a sufficiently larger value while keeping Δ fixed will make the IS of two Chua's oscillators unattainable no matter how small we choose $||B||$. This is illustrated in Figure 4(b), panel(ii), where we can see that for $r = 2$, the error $\mathbf{e}$ will never become equi-attractive in the large even if we select a matrix B with a very small norm (e.g., in (b), we have taken $B = -0.01I$ and $\Delta = 0.002$). This suggests that there is a minimum threshold value for $||B||$ such that if we choose a matrix B whose norm is below that threshold value, equi-attractivity property will fail to hold.

Finally, Theorem 1 and Corollary 1 predict that the smaller the impulse duration Δ, the better the performance of synchronization toward larger delays (i.e., the synchronization error is less sensitive to larger delays). We verify this by reconsidering the above example where r, Δ and B were chosen to be 0.003, 0.002 and $-I$, respectively. We found out that the synchronization error is not equi-attractive in the large in this case. However, reducing the impulse duration Δ to 0.001 while keeping the other parameters fixed forces (26) to become uniformly equi-attractive in the large (see Figure 4(c)).

IV. Conclusion

We have established in this paper that the solutions to autonomous impulsive systems subject to bounded delay in their impulses are relatively insensitive to these delays and remain uniformly equi-attractive in the large. There are, however, two outstanding issues that remain uninvestigated. The first issue is concerned with examining the robustness of IS to channel noise and parameter mismatches, while the second issue is concerned with demonstrating how implementable this synchronization technique is to secure communication. Our main objective in the latter issue is to replace classical low-dimensional chaos generators (such as Chua's oscillator) by time-delay chaotic systems [4] which are known to generate extremely complex hyper-chaotic signals.

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