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Uniform Distribution of Objects in a Homogeneous Field: Cities on a Plain

The uniform distribution of objects in a homogenous field is familiar in all kinds of circumstances - atoms in liquids[1] , fibrils in cells[4], hairs on insects[5,6] and mammals[7], stomata on leaves[8], trees in forests[9~o] and cities on plains[11,12]. In many of these circumstances the relative positions of the objects may be described by a radial distribution function. In what follows we shall show that such a function may be used to study the distribution of cities on a plain.

Fig. 1 is a map of part of the Spanish Plateau lying south-east of Madrid and bounded by latitudes 0° 53' W, 3° 41' W and longitudes 38° 48' N, 40º 16' N. This area is homogeneous in climate, physiography, transportation, population density and economy. The towns are represented by circles whose radius is an approximate measure of physical extent[13]. We have chosen for analysis the 40 mile square centered at latitude 2° 30' W and longitude 39° 47' N which is especially homogeneous in town size and density.

The radial distribution function g(R) is defined as the probability that a structure will be in a unit area a distance R away from an arbitrary structure, divided by the number probability density. It may be computed for the towns lying in the square in Fig. 1 in the following way. Place each town in turn at the origin and surround it by a series of concentric circles spaced at a distance of $\Delta R = 1$ mile. Determine the area, $K_1(R)$ (lying wholly within the square in Fig. 1) of the annulus whose boundaries are at distances of $(R - \Delta R/2)$ and $(R + \Delta R/2)$ from town i. Let the number of towns lying within this annulus be $N_1(R)$. If N is the total number of towns in the square and K is the area of the square, then

$$g(R) = \frac{N(R)}{\rho K(R)}$$

where

$$N(R) = \sum_{i=1}^{N} N_1(R)$$

$$K(R) = \sum_{i=1}^{N} K_1(R)$$

$$\rho = N / K$$

In Fig. 2 the computed distribution function is indicated by the dashed curve drawn through the crosses which represent the points at which the determination was made; the radial distribution function for a random distribution is depicted by the dashed-dot line in Fig. 2.

The diminished density near the origin and the subsequent damped oscillations in density are characteristic of distributions in which objects are fundamentally randomly dispersed, but are subjected to repulsive interactions which tend to diminish the probability of finding two objects close to each other. If the distribution is formed by a dynamic process that guarantees that the distribution is maximally random subject to the constraints of the interparticle interactions, then the theoretical concepts developed in the study of the statistical mechanics of equilibrium liquids may be applied.

An example is useful. Assume that the towns are randomly distributed subject only to the constraint that no two town centres may be closer than a distance, **d**. Similar models have been proposed as prototypes for spatial interaction and pattern formation in biological systems[6,7,9]. Because the resulting distribution also arises in the hard-core model of classical liquids at equilibrium[2,3], the radial distribution functions for such distributions have been intensively studied. The results are tabulated[3] for different relative packing densities, ρ', where ρ' is the number density divided by the maximum close packed hard-core density, and is given by

$$\rho' = \frac{\sqrt{3}}{2} \rho d^2$$

The best agreement is given when $\rho' = 0.5$, which corresponds to a hard-core town diameter of $d = 3.46$ miles (solid line in Fig. 2). Although there is qualitative agreement, the hard-core interaction is too stringent to give detailed quantitative agreement with the experimental results. A more realistic interaction between towns would be graded, in which the probability of finding two towns close together would be low, but would continuously increase as the distance between the towns increased. This is what would be expected from the geographic theory of central places[11,12]. Lawrence, in a study of hair placement in *Oncopeltus,* has hypothesized the existence of a very similar repulsive interaction which leads to the uniform distribution of hairs[6].

No matter what system is considered, this hypothesized repulsive field plays much the same role as the intermolecular potential in the theory of liquids. If the distribution of structures is maximally random subject to the constraints imposed by this repulsive field, then the equations developed in statistical mechanics may be applied. It is possible to compute theoretical radial distribution curves for a given hypothesized field (ref. 1, chapter 5) and also to determine, approximately, the interaction between structures if the radial distribution function is known (ref. 1, page 80). Future work may characterize the hypothesized repulsive fields and explore their physical, biological and social origins in those diverse disciplines where uniform distributions are observed.

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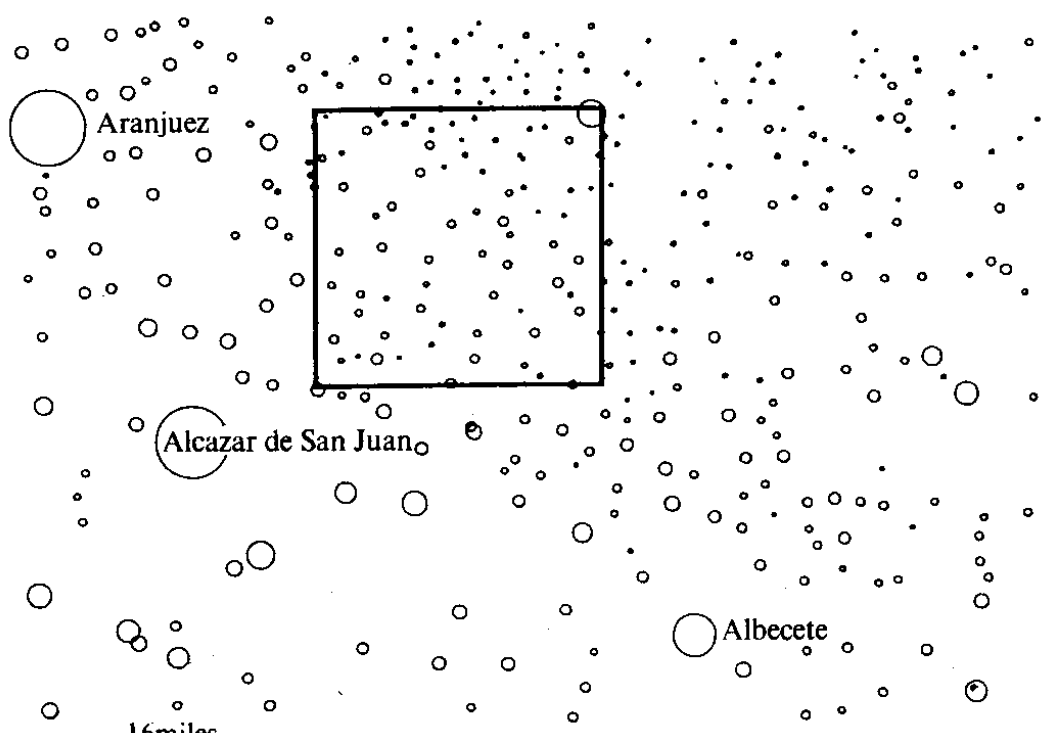

Fig. 1 A map of part of the Spanish Plateau lying south-east of Madrid, bounded by latitudes 0° 53′ W, 3° 41′ W and longitudes 38° 48′ N, 40° 16′ N. The radial distribution function of the cities lying in the square has been computed as described in the text. The locations of the towns were taken from the US Army Map Service topographic sheets of the Iberian Peninsula, 1 : 250,000, dated 1944.

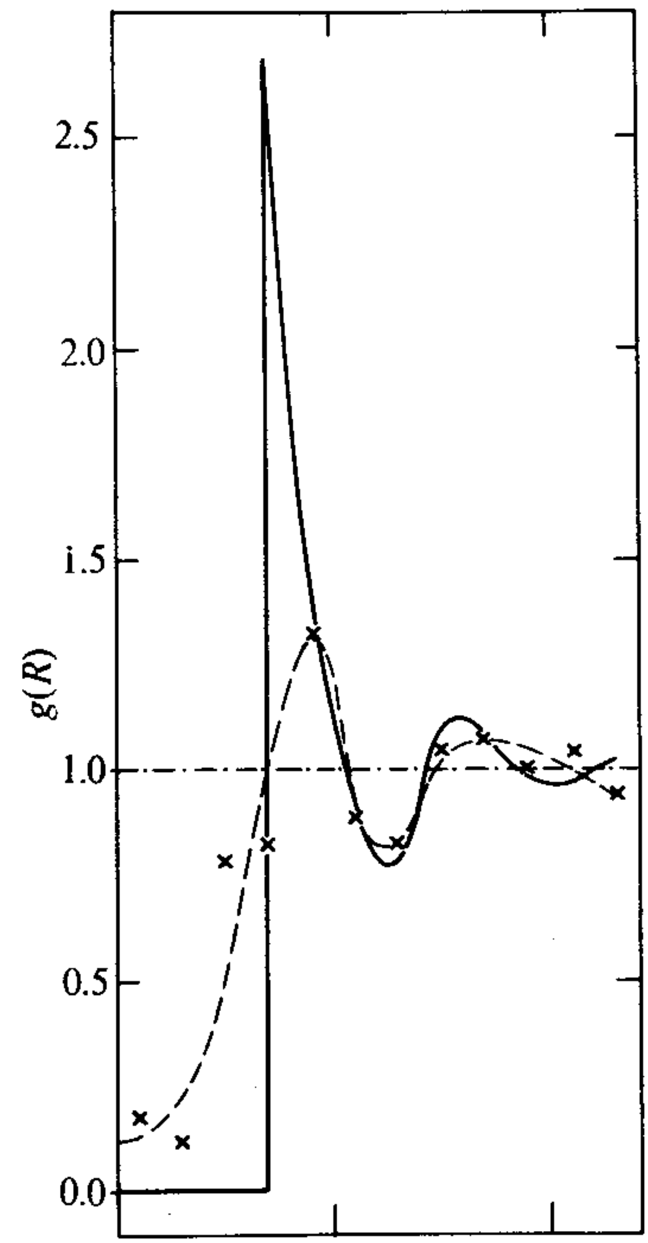

Fig. 2 The radial distribution function computed for the towns in the square in Fig. 1 (- - -), for a totally random distribution (—.—.—), and for a random distribution of hard-core disks with $\bar{\rho}=0.5$ and $d=3.46$ miles (——).