Physica D 40 (1989) 218–234
North-Holland, Amsterdam

SYMBOLIC DYNAMICS AND SKELETONS OF CIRCLE MAPS

Wan-Zhen ZENG and Leon GLASS
Department of Physiology, McGill University, Montreal, Quebec, Canada, H3G 1Y6

Received 27 January 1989
Revised manuscript received 23 June 1989
Accepted 16 July 1989
Communicated by R.M. Westervelt

A circle map maps the circumference of a circle into itself. Such maps are of physical and biological interest because they arise naturally in many circumstances, as for example the periodic forcing of limit cycle oscillators. The canonical example of circle maps is $f(x) = Tx + \tau + A \sin 2\pi x$ (mod 1) where A and τ are two constants. T is an integer that gives the topological degree of f. Cycles that contain extremal points of f are stable, and are called superstable cycles. The locus of superstable cycles in (A, τ) parameter space is called the skeleton. The critical points of f ($f'(x) = 0$) and the discontinuous points of f ($f(x) = 1$), break f into monotonic branches. In this paper we propose a novel notation for symbolically representing the branches of $f(x)$ and discuss the symbolic sequences for the skeleton. Simple rules based on continuity of periodic orbits as parameters change, allow us to construct the skeleton associated with all periodic orbits for the circle maps with $T = 0$, $0 < A \leq 1$, $\tau = 0$ and $T = 1$, $0 < A \leq 1$, $\tau = 0$. Recursion formulae to compute the numbers of periodic orbits for $T = 0$ and $T = 1$ at $\tau = 0$ are given.

1. Introduction

The dynamics arising from periodic stimulation of strongly attracting limit cycle oscillations can be approximated by one-dimensional maps of the circle f: $S^1 \to S^1$ [1–5]. For example, such maps arise in studies of the periodic forcing of biological [6–8] and chemical [9] oscillators by brief stimuli. Circle maps also arise in studies of dynamics of laser systems [10–12]. Although the detailed quantitative form of the maps differ from case to case, topological features of the maps from different systems are often the same. Accordingly intensive studies have been undertaken to identify certain universal features of the bifurcations of circle maps that are independent of the precise quantitative details. Significant advances have been made in the description of the bifurcation and scaling characteristics for invertible one-dimensional diffeomorphisms of the circle and particularly for parameter values at which such maps lose invertibility by developing cubic nonlinearities [13–16]. However, as practical applications accumulate, it is clear that many interesting experiments and theoretical examples arise for parameter values at which the map is not invertible [6–11]. Although there has been some analysis of those cases [2–12, 17–24], a complete theory does not exist. The main motivation for the current paper is to develop analytic techniques to study the global topological structure of the bifurcations of noninvertible circle maps.

One of the primary techniques to analyze noninvertible maps with extremal points is to consider the periodic orbits containing extremal points [25]. Such a periodic orbit is guaranteed to be stable,

0167-2789/89/$03.50

and is called a superstable orbit. Quite early, important theoretical methods were developed describing topological features of superstable cycles in one-dimensional maps with a single extremum [25–27]. Periodic orbits are represented by giving sequences of symbols corresponding to whether the iterate of the map is to the right or left of the extremal point [25–27]. This approach is called symbolic dynamics. Symbolic dynamics has also been extended to the analysis of cubic maps [28–32].

In maps with a two-dimensional parameter space, the locus of points associated with a given superstable cycle has been called a bone [33] and the union of bones is called the skeleton [19, 33]. The bones are ordered in the two-dimensional parameter space and two bones associated with the same critical point cannot intersect [19]. Numerical computation of the skeleton is straightforward and has been reported in a number of different contexts [4, 5, 8, 20–22, 34]. A detailed theory of the symbolic dynamics for the skeleton of the bimodal map was developed by MacKay and Tresser [33]. However, there are still few results pertaining to the symbolic dynamics of circle maps [5, 17, 35].

In this paper we study the circle map

$$f(x) = Tx + \tau + A\sin 2\pi x \pmod 1, \tag{1}$$

where T is an integer giving the topological degree of the map and A and τ are positive real numbers. Although most studies have dealt with this map with $T = 1$ [13–16, 19, 20, 23, 24], maps with topological degree 0 are likewise important in studies of forced oscillators [1–5] and laser systems [10–12].

This paper is based on the following idea. The skeleton is invariant under translations of $\tau = \tau_0 \pm N$ where N is an integer (since the map is taken modulo 1). As A increases from $1/2\pi$, bones of the skeleton can be continued and intersect the symmetry lines $\tau = \pm N$ where N is an integer. Consequently, a great deal of information about the global structure of the skeleton can be potentially derived from a knowledge of the symbolic sequences at $\tau = 0$, if it is known how the bones of the skeleton intersecting $\tau = 0$ can be continued in (A, τ) parameter space. Indeed it may be possible to completely reconstruct the global structure of the skeleton of eq. (1) based solely on analytical arguments for the symbolic dynamics along $\tau = 0$ combined with continuity arguments. However, the computations rapidly become difficult when A is large because of the large number of symbols needed to characterize orbits, and the complex transitions of symbolic sequences that occur along the skeleton. Consequently, only partial results have been obtained thus far. Our approach to this problem is one of physicists, rather than mathematicians. We have used numerical simulations to help discover rules for constructing the skeleton over limited regions of parameter space in eq. (1). Plausibility arguments to help support these rules are given. Rigorous proofs suitable for a mathematician have not been developed and consequently the rules we give should be considered conjectures.

In section 2 we develop the key theoretical concepts needed to discuss the symbolic dynamics of eq. (1). A simple symbolic notation for superstable cycles and rules for ordering the superstable cycles are given. The concept of rotation number is introduced and its connection with the symbolic sequences is derived. In section 3 we discuss admissible sequences of superstable cycles in parameter space for circle maps of degree 0 and 1 when $\tau = 0$. In section 4 we give results pertaining to the symbolic dynamics of the skeleton of eq. (1). This problem involves extension of the ordered sequences along symmetry lines found in section 3, to the complete space of two parameters. A number of rules have been found for this procedure. In section 5 we show that in maps with $T \neq 1$ the rotation number is not invariant along some bones of the skeleton. The conclusions are given in section 6.

Appendix A gives methods to determine admissible sequences for circle maps with $T = 0$, $\tau = 0$ based on the λ-expansion method [27]. Techniques developed earlier [31, 32, 36–42] to determine the number of stable cycles in quadratic and cubic maps are adapted for circle maps with $\tau = 0$ in appendix B.

2. Symbolic sequence of the circle map

2.1. *The lift and topological degree of circle maps*

The lift and topological degree of circle maps are discussed in ref. [22]. Here we give the definitions and refer the reader to the earlier paper for a full discussion.

Consider the continuous map F: $\mathrm{R} \to \mathrm{R}$ with the symmetry $F(x+1) = F(x) + T$ for all $x \in \mathrm{R}$. By considering $f = F$ (mod 1) we restrict the function F to the circle S and thus define a map f: $\mathrm{S}^1 \to \mathrm{S}^1$. The function F is called the *lift* of f, and T is the *topological degree* of f.

2.2. *The symbolic sequence*

The circle map, eq. (1), with degree T has two critical points X_i and $X_{i'}'$,

$$X_i = \frac{1}{2} - \frac{1}{2\pi}\cos^{-1}\left(-\frac{T}{2\pi A}\right),$$

$$X_{i'}' = \frac{1}{2} + \frac{1}{2\pi}\cos^{-1}\left(-\frac{T}{2\pi A}\right),$$

where the subscripts $i = F(X_i) - f(X_i)$ and $i' = F(X_{i'}') - f(X_{i'}')$. The two points X_i and $X_{i'}'$ divide the interval [0,1] into three subintervals $(0, X_i)$, $(X_i, X_{i'}')$ and $(X_{i'}', 1)$ denoted L, M and R, respectively. Any point x of the map (1) except the critical point can therefore be associated with a symbol L_j, M_j, or R_j, where the subscript j is given by

$$j = F(x) - f(x). \tag{2}$$

In fig. 1 we show typical circle maps and the associated symbols. The subscript j changes at

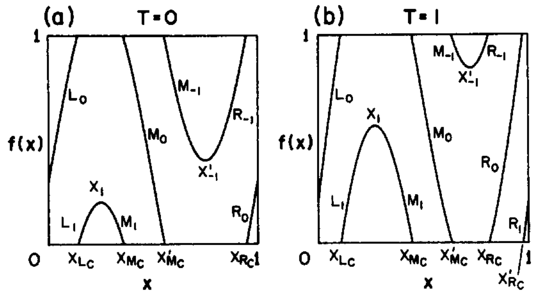

Fig. 1. Circle maps for (a) $T = 0$, $A = 0.9$, $\tau = 0.30$ and (b) $T = 1$, $A = 1.1$, $\tau = 0.2$.

discontinuous points such as $X_{\mathrm{M_c}}$, $X_{\mathrm{L_c}}$ in fig. 1. In what follows if $j = 0$ we will usually omit the subscript of the symbols.

By this definition any periodic orbit of period n, $x^1, x^2, x^3, \ldots, x^n$, can be represented by a symbolic sequence consisting of n symbols P^1, $\mathrm{P}^2, \mathrm{P}^3, \ldots, \mathrm{P}^n$. A superstable cycle is related to a symbolic sequence that starts and ends at either X_i or $X_{i'}'$. Because of the symmetry of the circle map $f(1-x, 1-\tau) = -f(x, \tau)$, throughout this study we primarily concentrate on the superstable orbits starting from X_i.

2.3. *Ordering of the symbolic sequence*

Before discussing the ordering of the symbolic sequence we give two definitions [5,6]. A symbolic sequence is *admissible* if it corresponds to a stable periodic orbit of a map. The parity of a symbolic sequence is determined by the number of points of the sequence that fall in region of the map with negative slope. The *parity* of a symbolic sequence is even, if the symbol M_j appears an even number of times; otherwise the parity of the sequence is odd.

In ref. [25], Metropolis, Stein and Stein gave rules to order admissible sequences of quadratic maps. Since the quadratic map has a single extremum there are only two symbols needed to construct symbolic sequences. More recent studies have dealt with symbolic dynamics of cubic maps using three symbols [28–32]. The situations considered here are more complicated since we deal with the symbolic dynamics of more than three symbols. Let

$$\mathrm{L}_{j_1} < \cdots < \mathrm{L}_{j_n} < \mathrm{M}_{k_1}$$
$$< \cdots < \mathrm{M}_{k_m} < \mathrm{R}_{i_1} < \cdots < \mathrm{R}_{i_l}, \tag{3}$$

with the subscripts

$$j_1 < \cdots < j_n, \; k_1 > \cdots > k_m \text{ and } i_1 < \cdots < i_l.$$

The ordering of subscripts of the M-branch is inverted since the M-branch has a negative slope [25].

It is possible to order two sequences P_α and P_β. Suppose $\mathrm{P}_\alpha = \mathrm{P}^*\sigma\ldots$, $\mathrm{P}_\beta = \mathrm{P}^*\mu\ldots$ where P^* is the common part of the sequences P_α and P_β, and $\sigma \neq \mu$. If P^* is even, then $\sigma > \mu$ gives $\mathrm{P}_\alpha > \mathrm{P}_\beta$ and $\sigma < \mu$ gives $\mathrm{P}_\alpha < \mathrm{P}_\beta$; if P^* is odd, then $\sigma > \mu$ gives $\mathrm{P}_\alpha < \mathrm{P}_\beta$ and $\sigma < \mu$ gives $\mathrm{P}_\alpha > \mathrm{P}_\beta$. As an example consider the sequences $\mathrm{P}_\alpha = \mathrm{P} = X_0\mathrm{R}_{-1}\mathrm{R}_{-1}\mathrm{LM}_{-1}$ and $\mathrm{P}_\beta = \mathrm{P}' = X_0\mathrm{R}_{-1}\mathrm{M}_{-1}\mathrm{LM}_{-1}$ shown in fig. 2. Since $\mathrm{P}^* = X_0\mathrm{R}_{-1}$ which is even and $\mathrm{R}_{-1} > \mathrm{M}_{-1}$, we have $\mathrm{P} > \mathrm{P}'$.

The ordering of the symbolic sequences can be used to impose an ordering on the points in a cycle. Consider the periodic orbit $x^1, x^2, \ldots, x^n$ and its associated symbolic sequence $\mathrm{P}^1, \mathrm{P}^2, \ldots, \mathrm{P}^n$. Call $\underline{\mathrm{P}}^i$ the sequence $\mathrm{P}^i\mathrm{P}^{i+1}\ldots\mathrm{P}^n$. Then $x^i > x^j$ if $\underline{\mathrm{P}}^i > \underline{\mathrm{P}}^j$ and $x^i < x^j$ if $\underline{\mathrm{P}}^i < \underline{\mathrm{P}}^j$. This observation is crucial to construct the skeleton in section 4. As an example, consider the sequence $\mathrm{P} = X_0\mathrm{R}_{-1}\mathrm{R}_{-1}\mathrm{LR}_{-1}$ in fig. 3. $\underline{\mathrm{P}}^2 = \mathrm{R}_{-1}\mathrm{R}_{-1}\mathrm{LR}_{-1}$, $\underline{\mathrm{P}}^3 = \mathrm{R}_{-1}\mathrm{LR}_{-1}$, and $\underline{\mathrm{P}}^5 = \mathrm{R}_{-1}(X_0)$. For $\underline{\mathrm{P}}^5$ the symbol (X_0) is enclosed in parentheses since it is the next symbol that would be encountered. Since $\mathrm{R}_{-1} > X_0 > \mathrm{L}$ we have $\underline{\mathrm{P}}^2 > \underline{\mathrm{P}}^5 > \underline{\mathrm{P}}^3$ and hence $x^2 > x^5 > x^3$.

2.4. *Rotation number*

The rotation number counts the average rotation of a point under subsequent iterates of a circle map. For a circle map with $T = 1$ the rotation number of a periodic orbit n passing through

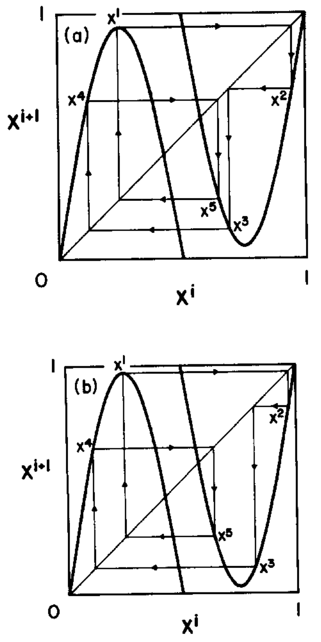

Fig. 2. Symbolic sequences (a) $\mathrm{P}' = X_0\mathrm{R}_{-1}\mathrm{M}_{-1}\mathrm{LM}_{-1}$ and (b) $\mathrm{P} = X_0\mathrm{R}_{-1}\mathrm{R}_{-1}\mathrm{LM}_{-1}$. In parameter space $\tau > 0$, P′ and P connect to each other and form a bone. $T = 0$, $\tau = 0$; for (a) $A = 0.9469$, for (b) $A = 0.9698$.

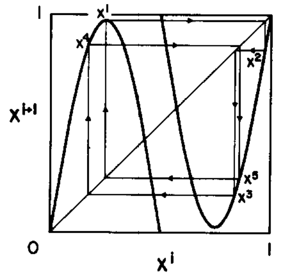

Fig. 3. Symbolic sequence $X_0\mathrm{R}_{-1}\mathrm{R}_{-1}\mathrm{LR}_{-1}$ for $T = 0$ circle map with $\tau = 0$, $A = 0.9734$.

point x_0 is given by

$$\rho = \frac{F^n(x_0) - x_0}{n}. \tag{4}$$

For circle maps with degree different from 1 the rotation number of a periodic orbit may depend on the initial point using this definition. For example, using eq. (4), the periodic orbit $1/7 \to 2/7 \to 4/7 \to 1/7$ in the map $f(x) = 2x \pmod 1$ has a rotation number of 1/3, 2/3 or 4/3 depending on whether x_0 is chosen as 1/7, 2/7 or 4/7, respectively.

An alternative definition for the rotation number for circle maps can be based on the symbolic sequence. Let

$$\mathrm{P} = \mathrm{P}_{i_1}^1 \mathrm{P}_{i_2}^2 \dots \mathrm{P}_{i_n}^n,$$

where the subscripts i_j represent the subscripts on the symbols using eq. (2), be a symbolic sequence of a period n orbit of the circle map. The rotation number is

$$\rho = \frac{1}{n} \sum_{j=1}^{n} i_j. \tag{5}$$

This definition is equivalent to an earlier formulation [22]. It is equivalent to eq. (4) for degree 1 circle maps. This definition gives the rotation number of the periodic orbit $1/7 \to 2/7 \to 4/7 \to 1/7$ considered above as 1/3, independent of the initial point.

3. Admissible sequences

This paper is directed towards a determination of the skeleton of circle maps, eq. (1), based on a knowledge of the admissible sequences at $\tau = 0$. In this section we consider techniques used to determine the admissible sequences, and the number of admissible sequences at $\tau = 0$ for eq. (1) with $T = 0$, $T = 1$. The techniques used are extensions of methods developed originally for the study of quadratic [27] and cubic [32, 42] maps. We summarize the results in this section and give additional technical details in the appendices.

The admissible sequences at $\tau = 0$ can be determined numerically from the solutions of the equation

$$X_j = f^n(X_j, A), \quad j = 0, 1, \tag{6}$$

where X_j is the critical point of the map. The values of A determined numerically and the associated symbolic sequences are shown in fig. 4 for the $T = 0$ map and figs. 5, 6 for the $T = 1$ map.

3.1. *$T = 0$ (fig. 4)*

For $0 \le A \le 1/2$ the periodic orbits originating from X_0 are identical to those found in the quadratic map [25]. Consequently the symbolic sequences here are well known from previous studies and are not repeated here [25, 27]. For $1/2 \le A \le 1$ the admissible sequences can be determined by adopting methods introduced by Derrida, Gervois and Pomeau (DGP) for the quadratic map [27]. this is presented in appendix A. The number of periodic orbits for $0 \le A \le 1$ can be determined analytically without an enumeration of the admissible sequences. This can be done in two different ways by adopting recursion formulae as well as group theoretic methods. The computations using recursion formulae are given in appendix B.

3.2. *$T = 1$ (figs. 5 and 6)*

At $A = 0.732$ the circle map is an onto map. For $0 \le A \le 0.73$ (fig. 5) the periodic orbits originating from X_0 are identical to those found in the cubic map [29, 32] and consequently the symbolic sequences are known from previous studies [28–32]. We have not found analytic techniques to determine the admissible sequences for $0.732 \le A \le 1$ and the results here are known from numerical results using eq. (6). The total number of periodic orbits found agree with analytic computations using recursion and group theoretic techniques (appendix B).

4. The skeleton of the circle map

We now discuss how the admissible sequences along the $\tau = 0$ axis extend to the whole parameter space. To study this problem it is necessary to consider the changes in the symbolic sequences

Period	Rotation Number	A	Symbolic Sequence
5	- 4/5	0.99945	$R_{-1}R_{-1}R_{-1}R_{-1}$
5	- 4/5	0.99849	$R_{-1}R_{-1}R_{-1}M_{-1}$
5	- 3/5	0.99768	$R_{-1}R_{-1}R_{-1}M$
4	- 3/4	0.99653	$R_{-1}R_{-1}R_{-1}$
5	- 3/5	0.99464	$R_{-1}R_{-1}R_{-1}L$
5	- 3/5	0.99240	$R_{-1}R_{-1}M_{-1}L$
4	- 3/4	0.99039	$R_{-1}R_{-1}M_{-1}$
5	- 3/5	0.98910	$R_{-1}R_{-1}M_{-1}M$
5	- 4/5	0.98815	$R_{-1}R_{-1}M_{-1}M_{-1}$
5	- 4/5	0.98702	$R_{-1}R_{-1}M_{-1}R_{-1}$
5	- 2/5	0.98612	$R_{-1}R_{-1}ML$
4	- 2/4	0.98503	$R_{-1}R_{-1}M$
5	- 2/5	0.98383	$R_{-1}R_{-1}MM$
5	- 3/5	0.98262	$R_{-1}R_{-1}MM_{-1}$
5	- 3/5	0.98049	$R_{-1}R_{-1}MR_{-1}$
3	- 2/3	0.97725	$R_{-1}R_{-1}$
5	- 3/5	0.97343	$R_{-1}R_{-1}LR_{-1}$
5	- 3/5	0.96978	$R_{-1}R_{-1}LM_{-1}$
5	- 2/5	0.96665	$R_{-1}R_{-1}LM$
4	- 2/4	0.95961	$R_{-1}R_{-1}L$
4	- 2/4	0.95800	$R_{-1}X'_{-1}L$
5	- 2/5	0.95058	$R_{-1}M_{-1}LM$
5	- 3/5	0.94693	$R_{-1}M_{-1}LM_{-1}$
5	- 3/5	0.94269	$R_{-1}M_{-1}LR_{-1}$
3	- 2/3	0.93870	$R_{-1}M_{-1}$
5	- 3/5	0.93529	$R_{-1}M_{-1}MR_{-1}$
5	- 3/5	0.93251	$R_{-1}M_{-1}MM_{-1}$
5	- 2/5	0.93070	$R_{-1}M_{-1}MM$
4	- 2/4	0.92905	$R_{-1}M_{-1}M$
5	- 2/5	0.92754	$R_{-1}M_{-1}ML$
5	- 4/5	0.92615	$R_{-1}M_{-1}M_{-1}R_{-1}$
5	- 4/5	0.92469	$R_{-1}M_{-1}M_{-1}M_{-1}$
5	- 4/5	0.92313	$R_{-1}M_{-1}M_{-1}M$
4	- 3/4	0.92108	$R_{-1}M_{-1}M_{-1}$
4	- 3/4	0.91333	$R_{-1}M_{-1}R_{-1}$
5	- 3/5	0.91126	$R_{-1}M_{-1}R_{-1}M$
5	- 4/5	0.90964	$R_{-1}M_{-1}R_{-1}M_{-1}$
5	- 4/5	0.90821	$R_{-1}M_{-1}R_{-1}R_{-1}$
5	- 1/5	0.90676	$R_{-1}MLL$
4	- 1/4	0.90521	$R_{-1}ML$
5	- 1/5	0.90351	$R_{-1}MLM$
5	- 2/5	0.90150	$R_{-1}MLM_{-1}$
5	- 2/5	0.89860	$R_{-1}MLR_{-1}$
3	- 1/3	0.89592	$R_{-1}M$
5	- 2/5	0.89321	$R_{-1}MMR_{-1}$
5	- 2/5	0.88990	$R_{-1}MMM_{-1}$
5	- 1/5	0.88723	$R_{-1}MMM$
4	- 1/4	0.88484	$R_{-1}MM$
5	- 1/5	0.88246	$R_{-1}MML$
5	- 3/5	0.87989	$R_{-1}MM_{-1}R_{-1}$
5	- 3/5	0.87746	$R_{-1}MM_{-1}M_{-1}$
5	- 2/5	0.87386	$R_{-1}MM_{-1}M$
4	- 2/4	0.86836	$R_{-1}MM_{-1}$
4	- 2/4	0.83160	$R_{-1}MR_{-1}$
2	- 1/2	0.81313	R_{-1}
2	- 1/2	0.75000	X'_{-1}
5	- 2/5	0.70745	$M_{-1}MM_{-1}M$
5	- 1/5	0.68393	$M_{-1}MML$
4	- 1/4	0.67888	$M_{-1}MM$
5	- 1/5	0.67357	$M_{-1}MMM$
3	- 1/3	0.66145	$M_{-1}M$
5	- 1/5	0.65092	$M_{-1}MLM$
4	- 1/4	0.64681	$M_{-1}ML$
5	- 1/5	0.64325	$M_{-1}MLL$
5	- 3/5	0.62858	$M_{-1}M_{-1}R_{-1}M$
5	- 3/5	0.61726	$M_{-1}M_{-1}M_{-1}M$
5	- 2/5	0.59410	$M_{-1}M_{-1}ML$
4	- 2/4	0.57950	$M_{-1}X'_{-1}M$
4	- 2/4	0.57935	$M_{-1}R_{-1}M$
5	- 2/5	0.55642	$M_{-1}R_{-1}ML$

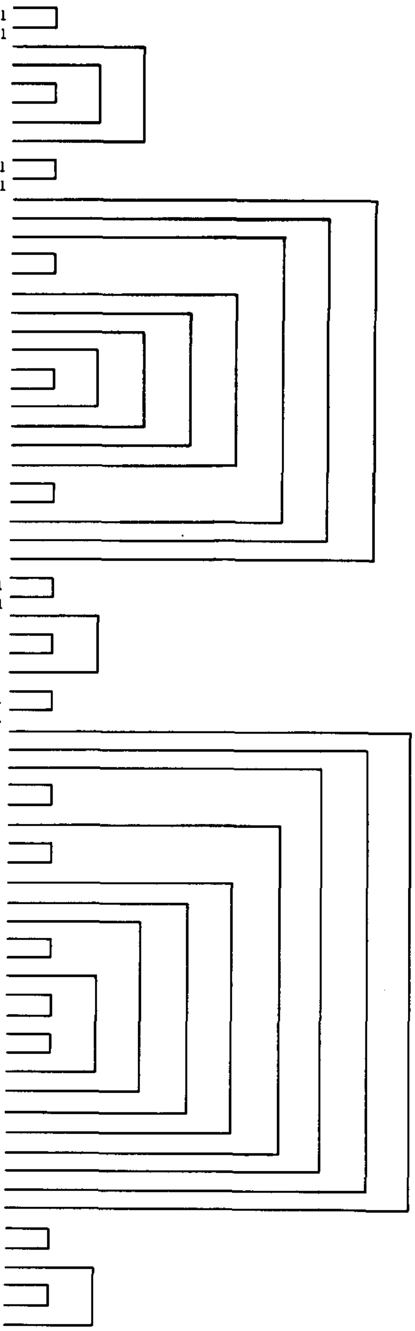

Fig. 4. Skeleton for $T = 0$ circle map, $\tau = 0$, with $1/2 < A < 1.0$. In this and following figures the numerically determined values of A associated with each superstable cycle are shown. The connections are given by rule 1.

(called *symbolic transitions*) along the bones of the skeleton. Several rules for connecting two different admissible sequences at $\tau = 0$ have been discovered. We only consider symbolic transitions for superstable cycles that contain X_j.

4.1. *Continuity rule*

Symbolic transitions occur when a point of a periodic orbit either crosses a discontinuous point (such as X_{L_c}, X_{M_c}, $X'_{M_c}, \ldots$, fig. 1) or the critical point $X'_{j'}$. In the case of a periodic point crossing a discontinuous point, the symbolic sequence has a change in two symbols; if a periodic point crosses the critical point $X'_{j'}$, there is a change in one symbol $M_{j'} \leftrightarrow R_{j'}$.

In a given cycle there may be a number of points all identified with the same symbol S. The largest of these is called the *leading maximum coordinate on* S and the smallest of these is called the *leading minimum coordinate on* S. For example x^2 (x^3) is the leading maximum (minimum) coordinate on R_{-1} in fig. 3. We can now state the continuity rule.

Continuity rule. Consider the symbolic transition $P \leftrightarrow P'$ where $P = P^1P^2 \ldots P^kP^{k+1} \ldots P^n$, and $P' = P^1P^2 \ldots (P^k)'(P^{k+1})' \ldots P^n$. P and P′ are identical except in the k-locus for transitions involving the critical point and in the k- and $(k+1)$-loci for transitions involving a discontinuous point. P is associated with the orbit $x^1, x^2, \ldots, x^n$ and P′ is associated with the orbit y^1, $y^2, \ldots, y^n$. If $P^k > (P^k)'$ then x^k is the leading minimum coordinate on branch P^k and y^k is the leading maximum coordinate on branch $(P^k)'$.

For example, the two periodic sequences in fig. 2 entail a single symbolic transition. Calling $P = X_0R_{-1}R_{-1}LM_{-1}$ and $P' = X_0R_{-1}M_{-1}LM_{-1}$ we have $P^3 = R_{-1}$ and $(P^3)' = M_{-1}$. Since $P^3 > (P^3)'$, x^3 is the leading minimum coordinate on R_{-1} in fig. 2b and y^3 is the leading maximum coordinate on M_{-1} in fig. 2a.

Period	Rotation Number	A	Symbolic Sequence
5	0	0.73181	RRRR
5	0	0.72997	RRRM
4	0	0.72790	RRR
5	0	0.72496	RRRL
5	0	0.72097	RRML
4	0	0.71731	RRM
5	0	0.71398	RRMM
5	0	0.70967	RRMR
3	0	0.70451	RR
5	0	0.69861	RRLR
5	0	0.69133	RRLM
4	0	0.67873	RRL
4	0	0.67535	RX'L
5	0	0.66044	RMLM
5	0	0.64965	RMLR
3	0	0.64161	RM
5	0	0.63408	RMMR
5	0	0.62363	RMMM
4	0	0.61230	RMM
4	0	0.55390	RMR
2	0	0.52472	R
2	0	0.40370	X'

Fig. 5. Skeleton for $T = 1$, $\tau = 0$ circle map with $0 < A < 0.73$. The symbolic sequences are the same as the cubic map. The connections are given by rule 3.

4.2. *$T = 0$ circle maps*

The construction of the skeleton for the $T = 0$ circle map for all symbolic sequences with $0 \le A \le 1$ can be carried out based on the following two rules.

Rule 1. Symbolic sequences with $1/2 \le A \le 1.0$, $\tau = 0$ connect to each other in pairs. There is one symbolic transition along each bone associated with a crossing of the critical point X'_{-1} and a change in one symbol $M_{-1} \leftrightarrow R_{-1}$. Consider the periodic orbit $x^1x^2 \ldots x^i \ldots x^k \ldots x^n$ associated with the admissible sequence $P = P^1P^2 \ldots P^i \ldots P^k \ldots P^n$ where $P^i = M_{-1}$ and x^i is the leading maximal coordinate on branch M_{-1}, $P^k = R_{-1}$ and x^k is the leading minimal coordinate on branch R_{-1}. If $\underline{P^{i+1}} < \underline{P^{k+1}}$, then the symbolic transition happens at P^i, i.e. P connects to $P' = P^1 \ldots (P^i)' \ldots P^k \ldots P^n$ where $(P^i)' = R_{-1}$. If $\underline{P^{i+1}} > \underline{P^{k+1}}$, then the symbolic transition happens at P^k, i.e. P connects to $P' = P^1 \ldots P^i \ldots (P^k)' \ldots P^n$ where $(P^k)' = M_{-1}$.

The connections of fig. 4 are all consistent with this rule. As an example consider a cycle $x^1x^2x^3x^4x^5$ associated with $P = X_0R_{-1}R_{-1}LM_{-1}$

Period	Rotation Number	A	Symbolic Sequence	
5	4/5	0.97771	$L_1L_1L_1L$	$\rightarrow X_1L_1L_1LR_1 \rightarrow X_1L_1LR_1R_1 \rightarrow X_1LR_1R_1R_1 \rightarrow X_0R_1R_1R_1R_1 \rightarrow$ 4/5
5	3/5	0.97301	L_1L_1LR	
5	3/5	0.96621	L_1L_1LM	
4	3/4	0.96247	L_1L_1L	$\rightarrow X_1L_1LR_1 \rightarrow X_1LR_1R_1 \rightarrow X_0R_1R_1R_1$ – 3/4
5	3/5	0.95966	L_1L_1LL	– L_1LR_1L
5	2/5	0.95801	L_1LRR	
5	2/5	0.95440	L_1LRM	
4	2/4	0.95152	L_1LR	
5	2/5	0.94862	L_1LRL	
5	1/5	0.94614	$L_1LR_{-1}R$	
5	1/5	0.93896	$L_1LM_{-1}R$	
5	2/5	0.93675	L_1LML	
4	2/4	0.93434	L_1LM	
5	2/5	0.93200	L_1LMM	
5	2/5	0.92901	L_1LMR	
5	3/5	0.92733	L_1LM_1L	
3	2/3	0.92382	L_1L	$\rightarrow X_1LR_1 \rightarrow X_0R_1R_1 \rightarrow$ 2/3
5	3/5	0.92061	L_1LL_1L	$\rightarrow X_1LR_1L_1L \rightarrow X_1LR_1LR_1 \rightarrow X_0R_1R_1LR_1 \rightarrow$ 3/5
5	2/5	0.91918	L_1LLR	
5	2/5	0.91696	L_1LLM	
4	2/4	0.91549	L_1LL	$\rightarrow LR_1L$
5	2/5	0.91428	L_1LLL	$\rightarrow LR_1LL$
5	1/5	0.91349	LRRR	
5	1/5	0.91173	LRRM	
4	1/4	0.91015	LRR	
5	1/5	0.90840	LRRL	
5	0/5	0.90655	$LRR_{-1}R$	
5	0/5	0.90284	$LRM_{-1}R$	
5	1/5	0.90080	LRML	
4	1/4	0.89873	LRM	
5	1/5	0.89663	LRMM	
5	1/5	0.89380	LRMR	
5	2/5	0.89192	LRM_1L	
3	1/3	0.88865	LR	
5	2/5	0.88534	LRL_1L	
5	1/5	0.88323	LRLR	
5	1/5	0.87997	LRLM	
4	1/4	0.87752	LRL	
5	1/5	0.87527	LRLL	
5	0/5	0.87354	$LR_{-1}RR$	
5	0/5	0.86911	$LR_{-1}RM$	
4	0/4	0.86013	$LR_{-1}R$	
4	0/4	0.85786	$LX_{-1}'R$	
5	0/5	0.84868	$LM_{-1}RM$	
5	0/5	0.84416	$LM_{-1}RR$	
5	1/5	0.84228	LMLL	
4	1/4	0.83991	LML	
5	1/5	0.83724	LMLM	
5	1/5	0.83369	LMLR	
5	2/5	0.83078	LML_1L	
3	1/5	0.82831	LM	
5	2/5	0.82598	LMM_1L	$\rightarrow LMR_1L$
5	1/5	0.82311	LMMR	
5	1/5	0.81967	LMMM	
4	1/4	0.81688	LMM	
5	1/5	0.81385	LMML	
5	1/5	0.80564	LMRL	
4	1/4	0.80222	LMR	
5	1/5	0.79890	LMRM	
5	1/5	0.79487	LMRR	
5	2/5	0.79198	LM_1LL	
4	2/4	0.78412	LM_1L	
2	1/2	0.78216	L	$\rightarrow X_0R_1 \rightarrow$ 1/2
5	2/5	0.77431	LL_1LL	$\rightarrow X_1LLR_1L \rightarrow X_0R_1LR_1L \rightarrow$ 2/5
5	1/5	0.77136	LLRR	
5	1/5	0.76744	LLRM	
4	1/4	0.76408	LLR	
5	1/5	0.76042	LLRL	
5	1/5	0.75442	LLML	
4	1/4	0.75126	LLM	
5	1/5	0.74856	LLMM	
5	1/5	0.74543	LLMR	
3	1/3	0.74187	LL	$\rightarrow X_0R_1L \rightarrow$ 1/3
5	1/5	0.73859	LLLR	
5	1/5	0.73609	LLLM	
4	1/4	0.73294	LLL	$\rightarrow X_0R_1LL \rightarrow X_0RR_1L \rightarrow$ 1/4
5	1/5	0.73432	LLLL	$\rightarrow X_0R_1LLL \rightarrow X_0RR_1LL \rightarrow X_0RRR_1L \rightarrow$ 1/5

Fig. 6. Skeleton of $T = 1$, $\tau = 0$ circle map for $0.73 < A < 1$. The connections are given by rules 3, 4 and 5. The connections between the symbolic sequences and the sequences associated with the Arnold tongues are given by rule 5.

(fig. 2b). Here x^3 and x^5 are the leading minimum and maximum coordinates on R_{-1} and M_{-1}, respectively. Since $\underline{P}^4 = LM_{-1}$, $\underline{P}^6 = X_0$, the symbolic transition happens at $P^3 = R_{-1}$.

Rule 2. A symbolic sequence X_0MP^* for $0 \le A \le 1/2$, $\tau = 0$ connects to the symbolic sequence X_1LP^* for $1 < A$, $\tau = 0$ where P^* is common in both symbolic sequences and contains only M and L.

Allowed symbolic transitions. In order to determine the allowed symbolic transitions for the bones associated with the symbolic sequences at $\tau = 0$, $0 < A \le 1$, it is necessary to determine the allowed symbols. The smallest symbolic sequence at $\tau = 0$ is the period 2 orbit X_0M. The only allowed symbolic transitions are $X_0M \to X_0R \to X_1L$, where X_1L is the largest symbolic sequence at $\tau = 0$ (see fig. 7). From eq. (1) with $T = 0$ the first iterate of the extremal point (X_0 or X_1) is $\tau + A$. Since any orbit that starts from X_1 has to go to the L-branch, the condition $\tau + A < 1 + X_1 = 5/4$ must be satisfied. In this region of parameter space, the only allowed symbols are L, L_1, M_1, M, M_{-1}, R_{-1}, R and the only symbolic transitions that are possible are

$$M_j \leftrightarrow R_j, \quad LR \leftrightarrow L_1L, \quad MR \leftrightarrow M_1L,$$
$$ML \leftrightarrow M_{-1}R, \quad RL \leftrightarrow R_{-1}R, \quad X_1L \leftrightarrow X_0R,$$
$$j = -1, 0. \tag{7}$$

Remarks about rule 1. All bones of the skeleton for $\tau = 0$, $1/2 < A \le 1.0$, are contained in region I in fig. 7. At $\tau = 0$, $A = 1$, eq. (1) is an onto map. The symbolic sequences at $\tau = 0$ start $X_0M_{-1}\ldots$ or $X_0R_{-1}\ldots$, and consist only of the symbols L, M, M_{-1} and R_{-1}. The first symbolic transition that can arise (out of those in eq. (7)) are only $ML \to M_{-1}R$ and $M_{-1} \leftrightarrow R_{-1}$. However, R cannot be a symbol at $\tau = 0$. Consequently, if the first transition is $ML \to M_{-1}R$, then these two symbols must undergo further symbolic transitions, and the only possibilities are $M_{-1}R \to M_{-1}M$ or $M_{-1}R \to R_{-1}R \to R_{-1}M$. However, since neither

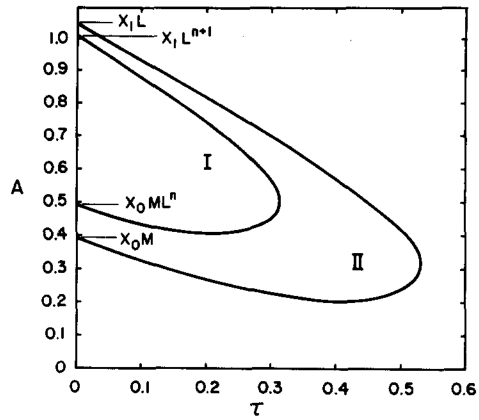

Fig. 7. Two of the bones of the $T = 0$ circle map. These two bones connect to the symbolic sequences at the lowest and largest values of A, $0 < A < 1/2$, at $\tau = 0$. These bones define regions I and II. In each region, symbolic sequences at $\tau = 0$ connect to each other in pairs.

of these transitions satisfy the continuity rule, the first symbolic transition must be $M_{-1} \leftrightarrow R_{-1}$. Assume that $P = P^1\ldots P^i\ldots P^n$ is an admissible sequence at $\tau = 0$ with $P^i = M_{-1}$ (or R_{-1}) and that the first transition happens at P^i. Further use of the continuity rule shows that no other symbolic transitions at P^k, $k \ne i$, are allowed. Consequently, $P' = P^1\ldots(P^i)'\ldots P^n$ with $(P^i)' = R_{-1}$ (or M_{-1}) is also an admissible sequence arising from $\tau = 0$ which is connected to P.

For the symbolic sequence P, a symbolic transition can happen either at the leading maximal coordinate on M_{-1} or the leading minimal coordinate on R_{-1}. The selection of the symbolic transition can be made by applying the continuity rule. As an example, $P = X_0R_{-1}R_{-1}LM_{-1}$ in fig. 2b is an admissible sequence connected to $X_0R_{-1}M_{-1}LM_{-1}$. Without the continuity rule, it would be possible for a transition to occur in $P^5 = M_{-1}$, so the connecting sequence would be $P' = X_0R_{-1}R_{-1}LR_{-1}$ (fig. 3). However, in P′, the last symbol R_{-1} is not the leading minimal coordinate, and consequently this transition is forbidden by the continuity rule.

Remarks about rule 2. In the region $\tau = 0$, $0 \le A \le 1/2$, the map eq. (1) is topologically equivalent to the quadratic map, the smallest and largest symbolic sequences are X_0M and X_0ML^n ($n \to \infty$), respectively. According to the allowed symbolic transitions they connect to X_1L and X_1L^{n+1} ($n \to \infty$) as shown in fig. 7. Although the number of symbolic transitions are limited, the symbolic transitions that can occur in region II in fig. 7 can be complicated. For example, we have the sequences

$$X_0\mathrm{MLMM} \to X_0\mathrm{RLMM} \to X_1\mathrm{LLMM},$$

$$\begin{aligned} X_0\mathrm{MLLM} &\to X_0\mathrm{M}_{-1}\mathrm{RLM} \\ &\to X_0\mathrm{R}_{-1}\mathrm{RLM} \\ &\to X_0\mathrm{RLLM} \to X_1\mathrm{LLLM}. \end{aligned}$$

Suppose P $= X_0$MLP* is associated with the periodic orbit $x^1x^2 \ldots x^n$ for $\tau = 0$, $0 < A \leq 1/2$. P$' = X_1$LL(P*)$'$ is associated with the periodic orbit $y^1y^2 \ldots y^n$ for $\tau = 0$, $A > 1$, where P* only contains M and L. Since x^2 is the leading maximum coordinate on M, the first symbolic transition on the bone connecting P and P$'$ is either X_0MLP* $\to X_0$RLP* or X_0MLP* $\to X_0$M$_{-1}$RP*. For the first case, the symbolic transitions on the bone connecting P and P$'$ are

$$\mathrm{P} = X_0\mathrm{MLP}^* \to X_0\mathrm{RLP}^* \to X_1\mathrm{LLP}^* = \mathrm{P}'.$$

The sequence P* in P only contains M and L, the first possible symbolic transitions for P* are M $\to$ R and ML $\to$ M$_{-1}$R. If a point x^j on M could move to R or M$_{-1}$, x^j has to reach the maximum value on M. By the continuity rule x^j has to pass a point where $f(x^j) = f(x^2)$. This means two periodic points would map into the same point. As an example see fig. 8b, where x^5 is the leading maximum coordinate on M. If x^5 could move to M$_{-1}$ it must pass a point c on M where $f(c) = f(x^2)$. Since this would change the length of cycle it is not allowed. Therefore the symbolic transitions between P and P$'$ cannot happen at P*.

If the first transition for P is ML $\to$ M$_{-1}$R, the symbolic transitions between P and P$'$ are P $=$ X_0MLP* $\to$ X$_0$M$_{-1}$RP* $\to \cdots \to X_0$R$_{-1}$R(P*)$'$ $\to X_0$RL(P*)$'$ $\to X_1$LL(P*)$'$. Here X_0R$_{-1}$R $\to$ X_0RL $\to X_1$LL are the last transitions since at $A > 1$, the symbolic sequence starts at x_1, the only transition from X_0 to X_1 is X_0R $\to X_1$L; in addition, a point on R is the leading minimum coordinate in the first transition and the leading maximum coordinate in the second transition. Therefore (P*)$'$ cannot contain R.

The possible symbolic transitions between X_0M$_{-1}$RP* and X_0R$_{-1}$R(P*)$'$ are ML $\leftrightarrow$ M$_{-1}$R, RL $\leftrightarrow$ R$_{-1}$R and M$_{-1} \leftrightarrow$ R$_{-1}$. Since P* only contains M, L and (P*)$'$ does not contain R, the symbolic sequence RP* undergoes the transitions ML $\to$ M$_{-1}$R, RL $\to$ R$_{-1}$R and M$_{-1} \leftrightarrow$ R$_{-1}$ along the bone has to take the transitions R$_{-1}$R $\to$ RL and M$_{-1}$R $\to$ ML in order to ensure that (P*)$'$ does not contain R. By noticing that R always follows M$_{-1}$ or R$_{-1}$, (P*)$'$ also cannot contain M$_{-1}$ and R$_{-1}$. Therefore (P*)$'$ = P*.

4.3. *T = 1 circle map*

Bones of the skeleton that extend to $A = 1/2\pi$ are called *primary bones*, fig. 9. These arise from

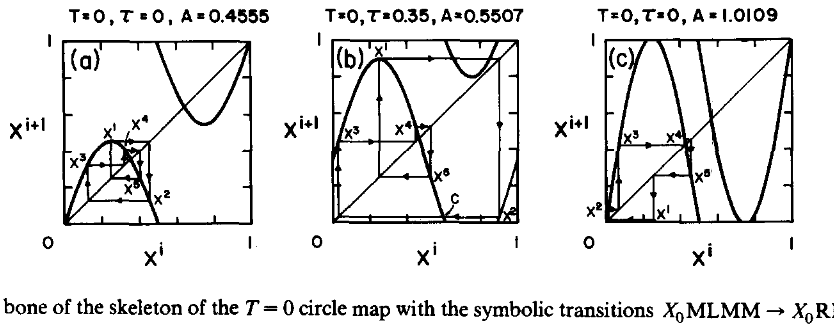

Fig. 8. Maps along a bone of the skeleton of the $T = 0$ circle map with the symbolic transitions X_0MLMM $\to X_0$RLMM $\to X_1$LLMM.

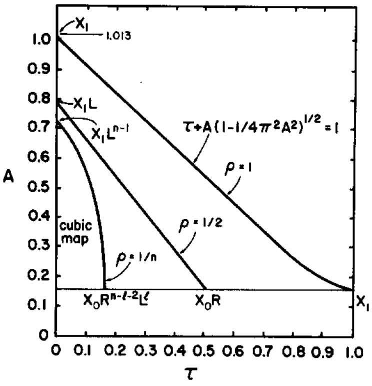

Fig. 9. Three bones of the skeleton for $T = 1$ circle map with rotation number $\rho = 1/n$ ($n \to \infty$), $\rho = 1/2$ and $\rho = 1$. In the region indicated, all symbolic sequences are identical to those of the cubic map.

the cubic inflection point when the circle map becomes noninvertible. For each rational rotation number there is one primary bone associated with each critical point. The remainder of the bones at $\tau = 0$ connect to each other in pairs. Rules 3 and 4 give the allowed symbolic transitions for all bones that are not primary bones. Rule 5 gives the symbolic sequences of the primary bones at $\tau = 0$ and specifies sequences of symbolic transitions allowed on them.

Rule 3. Suppose the periodic orbit $x^1x^2 \dots x^i \dots x^k \dots x^n$ is associated with the symbolic sequence $\mathrm{P} = \mathrm{P}^1 \dots \mathrm{P}^i \dots \mathrm{P}^k \dots \mathrm{P}^n$. P contains M_j or (and) R_j; say $\mathrm{P}^i = \mathrm{M}_j$ and $\mathrm{P}^k = \mathrm{R}_j$ ($j = -1, 0$). Let x^i be the leading maximal coordinate on M_j and x^k be the leading minimal coordinate on R_j. If $\underline{\mathrm{P}_{i+1}} < \underline{\mathrm{P}_{k+1}}$, then the symbolic transition happens at P^i with $\mathrm{M}_j \to \mathrm{R}_j$. If $\underline{\mathrm{P}_{i+1}} > \underline{\mathrm{P}_{k+1}}$ then the symbolic transition happens at P^k with $\mathrm{R}_j \to \mathrm{M}_j$.

Rule 4. If the admissible sequence $\mathrm{P} = \mathrm{P}^1 \dots \mathrm{P}^{i-1}\mathrm{P}^i \dots \mathrm{P}^n$ contains $\mathrm{P}^{i-1}\mathrm{P}^i = \mathrm{LM}_1$ with x^i the leading maximum coordinate on M_1, and P does not contain R_j or M_j ($j = -1, 0$), then P connects to $\mathrm{P}' = \mathrm{P}^1 \dots (\mathrm{P}^{i-1})'(\mathrm{P}^i)' \dots \mathrm{P}^n$ by the symbolic transitions in loci $j-1$ and j, $\mathrm{LM}_1 \to \mathrm{LR}_1 \to \mathrm{L}_1\mathrm{L}$.

Rule 5. (i) (Based on a rule for symbolic sequences for circle maps that give rigid rotations [5]). The rotation number $\rho = m/n$ (m and n have no common divisors) can be represented by a continued fraction,

$$\rho = \frac{m}{n} = \cfrac{1}{N_1 + \cfrac{1}{N_2 + \cfrac{1}{\cdots + \cfrac{1}{N_j}}}}.$$

The symbolic sequence associated with ρ at $\tau = 0$ is constructed recursively.

$$\mathrm{S}_1 = \mathrm{L}_1\mathrm{L}^{N_1 - 1},$$

$$\mathrm{S}_2 = \mathrm{S}_1\mathrm{L}\mathrm{S}^{N_2 - 1},$$

$$\mathrm{S}_j = \mathrm{S}_{j-1}\mathrm{S}_{j-2}, \quad \text{for } j \text{ odd} \geq 3,$$

$$\mathrm{S}_j = \mathrm{S}_{j-2}\mathrm{S}_{j-1}, \quad \text{for } j \text{ even} \geq 4.$$

The leading maximal coordinate in S_j on branch L_1 is X_1.

(ii) A symbolic sequence P at $\tau = 0$ is connected to P′ at $A = 1/2\pi$ by the symbolic transitions

$$\mathrm{L}_1\mathrm{L} \to \mathrm{LR}_1, \quad \mathrm{R}_1\mathrm{L} \to \mathrm{RR}_1,$$

where each symbolic transition is completely determined by the continuity rule and all transitions involve the leading minimal coordinate on the L-branch for $\mathrm{L}_1\mathrm{L}$ and $\mathrm{R}_1\mathrm{L}$. At $A = 1/2\pi$ the sequence P′ does not contain the symbol L_1.

Allowed symbolic transitions. The bone of the skeleton with the largest value of A is given by the superstable line with the rotation number $\rho = 1$ [7] (fig. 9). This means the critical point $X_1 = 1/2 - (1/2\pi)\cos^{-1}(1/2\pi A)$ is a fixed point along the skeleton. Therefore we have

$$F(X_1) = X_1 + \tau + A\sin 2\pi X_1 = 1 + X_1,$$

or

$$\tau + A(1 - 1/4\pi^2 A^2)^{1/2} = 1.$$

At $\tau = 0$, $A = (1 + 1/4\pi^2)^{1/2} \simeq 1.013$. Consider the skeleton associated with the symbolic sequences for $\tau = 0$ and $0 < A < 1.013$ (fig. 9). Since all symbolic transitions occur at the $\tau + A(1 - 1/4\pi^2 A^2)^{1/2} < 1$ parameter region, the allowed symbolic transitions are

$$\begin{array}{lll} \mathrm{M}_j \leftrightarrow \mathrm{R}_j, & \mathrm{MR}_1 \leftrightarrow \mathrm{M}_1\mathrm{L}, & \mathrm{LR}_1 \leftrightarrow \mathrm{L}_1\mathrm{L}, \\ \mathrm{ML} \leftrightarrow \mathrm{M}_{-1}\mathrm{R}_1, & \mathrm{RL} \leftrightarrow \mathrm{R}_{-1}\mathrm{R}_1, & \mathrm{RR}_1 \leftrightarrow \mathrm{R}_1\mathrm{L}, \\ X_1\mathrm{L} \leftrightarrow X_0\mathrm{R}_1, & & \end{array} \tag{8}$$

where $j = -1$, 0 and 1.

Remarks about rule 3. This rule is similar to rule 1 for degree 0 circle maps, and consequently similar arguments to those for rule 1 can be applied. Symbolic sequences for $\tau = 0$, $0 < A < 0.732\ldots$, where the map is topologically equivalent to the cubic map (see fig. 9) are connected to each other in pairs with only one transition $\mathrm{M} \leftrightarrow \mathrm{R}$. Consequently this rule completely specifies the symbolic transitions in the skeleton of the bimodal map $C_{p,q}(x) = x^3 + px + q$ associated with the symbolic sequences at $q = 0$. This map was studied by MacKay and Tresser [33].

Remarks about rule 4. If P does not contain R_j and M_j ($j = -1, 0$), the possible symbolic transitions in eq. (8) are $\mathrm{LM}_1 \leftrightarrow \mathrm{LR}_1$, $\mathrm{LR}_1 \leftrightarrow \mathrm{L}_1\mathrm{L}$ and $\mathrm{M}_1\mathrm{L} \leftrightarrow \mathrm{MR}_1$. If the transition $\mathrm{M}_1\mathrm{L} \rightarrow \mathrm{MR}_1$ occurred, the next transitions would be

$$\mathrm{M}_1\mathrm{L} \rightarrow \mathrm{MR}_1 \begin{array}{l} \nearrow \mathrm{RR}_1 \rightarrow \mathrm{RM}_1 \\ \searrow \mathrm{MM}_1 \end{array}$$

The transitions $\mathrm{RR}_1 \rightarrow \mathrm{RM}_1$ and $\mathrm{MR}_1 \rightarrow \mathrm{MM}_1$ do not obey the continuity rule. Since the symbolic sequences at $\tau = 0$ do not contain R_1, they have to follow the transitions $\mathrm{LM}_1 \leftrightarrow \mathrm{LR}_1 \leftrightarrow \mathrm{L}_1\mathrm{L}$ to connect to each other in pairs.

Remarks about rule 5. First we show that there are no periodic points on the M-branch for all primary bones. At $A = 1/2\pi$, the circle map is an invertible map. $0 < x < 1/2$ is associated with L_j, while $1/2 < x < 1$ with R_j, where j is defined in eq. (2). Since the map is a monotonic increasing function, for any point y on R_j we have $f(y) > f(X_j)$ where $X_j = 1/2$. For $A > 1/2\pi$ the map is noninvertible and $X_j = 1/2 - (1/2\pi)\cos^{-1}(1/2\pi A)$ is one of the critical points. If the symbolic transition $\mathrm{R}_j \rightarrow \mathrm{M}_j$ occurred by increasing A, then for any point z on M_j we have $f(z) < f(X_j)$. As A varies, a periodic point y would have to reach a value which satisfies $f(y) = f(X_j)$, in order to have the transition $\mathrm{R}_j \rightarrow \mathrm{M}_j$. This means that two periodic points y and X_j would be mapped to the same point. Since this would change the length of the cycle, it is not allowed and thus there are no symbols on the M-branch for primary bones.

If symbolic sequences on primary bones do not contain M_j, the transitions $\mathrm{M}_j \leftrightarrow \mathrm{R}_j$, $\mathrm{MR}_1 \leftrightarrow \mathrm{M}_1\mathrm{L}$ and $\mathrm{ML} \leftrightarrow \mathrm{M}_{-1}\mathrm{R}_1$ can be excluded from eq. (8). According to the continuity rule, the transition $\mathrm{RL} \rightarrow \mathrm{R}_{-1}\mathrm{R}_1$ requires that a periodic point y on R decreases to a minimum. From the previous argument we know that this cannot occur since y has to satisfy $f(y) > f(X_0)$. Therefore the transition $\mathrm{RL} \rightarrow \mathrm{R}_{-1}\mathrm{R}_1$ is not allowed. The symbolic transitions between the sequences P′ at $A = 1/2\pi$ and P at $\tau = 0$ are $\mathrm{RR}_1 \leftrightarrow \mathrm{R}_1\mathrm{L}$ and $\mathrm{LR}_1 \leftrightarrow \mathrm{L}_1\mathrm{L}$.

The association of a symbolic sequence for $\tau = 0$ with the symbolic sequence of the rigid circle map is based on the fact that P only contains L, L_1 and X_1. This can be shown by looking at the symbolic sequence P′ at $A = 1/2\pi$ which connects to P.

P′ with rotation number $\rho = m/n \le 1/2$ can be written as

$$\mathrm{P}' = X_0 \prod_{i=1}^{m} \left(\mathrm{R}^{A_i}\mathrm{R}_1\mathrm{L}^{B_i}\right), \tag{9}$$

where

$$A_i \geq 0, \quad B_i \geq 1 \qquad \text{for } i = 0 \text{ to } m,$$
$$|A_j - A_i| \leq 0, \quad |B_j - B_i| \leq 0 \quad \text{for } j \neq i,$$

and

$$\sum_{i=1}^{m} (A_i + B_i) + m + 1 = n.$$

A symbolic sequence with rotation number $\rho = 1 - m/n$ can be obtained from P′ by changing the symbols $\mathrm{R} \to \mathrm{L}_1$, $\mathrm{R}_1 \to \mathrm{L}$, $\mathrm{L} \to \mathrm{R}_1$ and $X_0 \to X_1$.

P′ connects to P by the symbolic transitions $\mathrm{RR}_1 \to \mathrm{R}_1\mathrm{L}$ and $\mathrm{LR}_1 \to \mathrm{L}_1\mathrm{L}$. Since P does not contain R_1, eventually P only contains L, L_1 and X_1.

The symbolic sequences at the cubic inflection in circle maps depend on the particular shapes of the maps. This can be verified by considering a different form of the circle map,

$$f(x) = \tau + x^3 \pmod 2,$$

where $x \in (-1, 1)$. Suppose the orbits start at $X_j = 0$, where X_j divides $(-1, 1)$ into two branches L_j and R_j. From computer simulation we determined that the symbolic sequence with rotation number $\rho = 2/9$ is $X_0\mathrm{RRR}_1\mathrm{LRRR}_1\mathrm{L}$. For the map eq. (1) at $T = 1$ and $A = 1/2\pi$ the corresponding symbolic sequence for $\rho = 2/9$ is $X_0\mathrm{RR}_1\mathrm{LLRRR}_1\mathrm{L}$. These two symbolic sequences differ by a single symbolic transition $\mathrm{RR}_1 \leftrightarrow \mathrm{R}_1\mathrm{L}$. Consequently, the symbolic sequences for degree 1 circle map for primary bones at the cubic inflection point are not universal.

5. Rotation number

Now we discuss the changes of rotation number along the bones of circle maps. Consider the circle map, eq. (1), and its lift $F(x)$.

First let us focus on the symbolic transition at $X_{\mathrm{L_c}}$ and look at the change of the rotation number when a periodic point x crosses $X_{\mathrm{L_c}}$. If $X^- = X_{\mathrm{L_c}} + 0^-$, where 0^- is a very small negative value, we have $F(X^-) = 1 + 0^-$ and $f(X^-) = 1^-$, the next iteration will be $F(f(X^-)) = T + \tau$ and $f(f(X^{-1})) = \tau$. The contribution to the rotation number is $\Delta\rho^- = (1/N)[F(X^-) - f(X^-) + F(f(X^-)) - f(f(X^-))] = T/N$, where N is the period of the orbit. If $X^+ = X_{\mathrm{L_c}} + 0^+$, then $F(X^+) = 1 + 0^+$, $f(X^+) = 0^+$ and $F(f(X^+)) = \tau$, $f(f(X^+)) = \tau$. The contribution to the rotation number is $\Delta\rho^+ = 1/N$. When the orbit passes through $X_{\mathrm{L_c}}$ the symbols of symbolic sequence are changed from LR_j to $\mathrm{L}_1\mathrm{L}$ and the change of corresponding rotation number is $(T-1)/N$. Therefore, only $T = 1$ circle maps maintain an invariant rotation number when the symbolic sequence changes as a point on a cycle crosses $X_{\mathrm{L_c}}$.

Similarly, when a point on a cycle crosses other discontinuous points of circle maps, the corresponding symbolic sequences will change, and the rotation number will also change for the $T \neq 1$ circle maps. Therefore only $T = 1$ circle maps maintain invariant rotation numbers along bones of the skeleton.

6. Conclusions

There are two main classes of problems associated with the dynamics of circle maps: (i) scaling aspects of the dynamics [13–16]; and (ii) topology of the bifurcations [2–5, 19–22, 25–27, 33]. Scaling arguments have played important roles in many areas of physics, and many beautiful results have been found relating to the dynamics of circle maps [13–16]. Our work has largely been motivated by experimental studies of periodically forced biological oscillators [6–8]. What is compelling in these experiments is the diversity of complex rhythms that can be found as stimulus parameters are varied, and it has been a continuing challenge to try to develop theoretical insight into the bifurcations in biological systems [2–5, 8]. This paper has shown that the topology of the bifurcations of circle maps can be largely determined using analytic methods based upon considerations of symbolic dynamics (24–28, 32–33] and continuity. Other recent work has used symbolic dynamics to compute the topological entropy and scaling properties at the intersection of Arnold tongues [43]. Thus, symbolic dynamics provides powerful analytic insights into the structure of periodic orbits of circle maps.

Detailed experimental study of bifurcations in physical and biological systems modeled by circle maps are difficult to carry out because of the very small sizes of the different phase-locking zones and their complex organization makes them difficult to observe. However, limited results, for example see the recent review [44], show that there may be discrepancies between observations and experiments. Such discrepancies may be accounted for by using higher dimensional maps, or by one-dimensional circle maps wtih discontinuities. Extending the techniques from one-dimensional continuous circle maps to these cases is a challenge for the future.

Acknowledgements

This research has been supported by the Natural Sciences Engineering and Research Council (Canada). We have benefitted greatly from conversations about this material with J. Bélair and J. Keener. M. Mackey made useful comments concerning the presentation. We thank S. James for typing the manuscript.

Appendix A

λ-expansion technique for the $T = 0$ circle map

Derrida, Gervois and Pomeau (DGP) developed a technique for analyzing periodic orbits of the quadratic map [27]. The DGP technique, in turn, is based on a number theoretic method for the representation of real numbers as series which was developed by Renyi [36]. Although rigorous proofs of these techniques have only been achieved for the quadratic map [27], the current work shows that generalizations to more complex situations also work.

The original DGP paper described the λ-expansion of a number $1 < x < 2$ with basis $1 < \lambda < 2$. Suppose

$$x_n = \sum_{i=0}^{n} \frac{C_i}{\lambda^i}, \tag{A.1}$$

where $C_{m+1} = 1$ if $x_m < x$, $C_{m+1} = -1$ if $x_m > x$. If the series terminates and $x = x_m$ for some finite value m then $C_{m'} = 0$ for all $m' > m$. Otherwise $\lim_{m\to\infty} x_m = x$. The notation for the set of coefficients characterizing a λ-expansion is $\{C_i\}$. The ordering of the λ-expansions $\{C_i\}$ and $\{D_i\}$ is achieved by considering the first different number in the two sets, say C_m and D_m. Then $\{C_i\} > \{D_i\}$ if $C_m > D_m$. The necessary conditions for $\{C_i\}$ to be a λ-expansion are

$$\begin{aligned}&(C_0, C_1, \dots) > (C_i, C_{i+1}, \dots),\\ &(C_0, C_1, \dots) > (-C_i, -C_{i+1}, \dots), \quad \text{for all } i > 0.\end{aligned} \tag{A.2}$$

Thus $(1, 1, -1, -1, -1, 0, 0, \dots)$ is not possible since, taking $i = 3$, $(1, 1, -1, -1, -1) < (1, 1, 1, 0, 0)$.

DGP showed that for a piecewise linear version of the quadratic map, called the tent map, the parameters of periodic orbits can be represented by finite λ-expansions. The condition for admissibility of symbolic sequences of the quadratic map then translates into the condition for λ-expansions of the tent map. This technique has been extended by Zeng [31, 32] to the cubic map.

We now consider a further extension to the $T = 0$ map with $1/2 < A < 1$ and $\tau = 0$. First we consider the λ-expansion of real numbers $X = \lambda/2$ with $2 < \lambda < 4$, which is defined by the two numbers X_n and C_n depending both on X and λ such that

$$X_n = \sum_{k=0}^{n} \frac{C_k}{\lambda^k}, \quad C_k = 0, \pm 1, \pm 2, \pm 1/2, \pm 3/2. \tag{A.3}$$

In eq. (A.3) the C_k are integers satisfying

$$(X - X_k)\lambda^{k+1} - 1/2 < C_{k+1} < (X - X_k)\lambda^{k+1} + 1/2$$

or the rational number given by

$$X - X_k = \frac{C_{k+1}}{\lambda^{k+1}}.$$

In this case $C_{k'} = 0$ for all $k' > k+1$. From the above it is possible to prove eq. (A.2) by induction,

$$|X - X_{k+1}| = \left| X - X_k - \frac{C_{k+1}}{\lambda^{k+1}} \right| < \frac{1}{2\lambda^{k+1}}. \tag{A.4}$$

Although C_k here are no longer necessarily $\pm 1, 0$ as defined by DGP, the necessary condition eq. (A.2) for eq. (A.3) to be λ-expansion still holds; eq. (A.4) serves as a key point to complete the proof.

Now we can associate the piecewise linear circle map $T = 0$ and $1/2 < A < 1$ with the λ-expansion discussed above. Consider the map

$$f(x) = \alpha\lambda x + \beta\lambda + \gamma, \tag{A.5}$$

where $x \in (0,4)$, $2 < \lambda < 4$ and α, β and γ depend on the branch L, M, M_{-1} and R_{-1} in such a way that

$$\begin{aligned}
&\mathrm{L}: && \alpha = 1, && \beta = 0, && \gamma = 0, && 0 < x \le 1,\\
&\mathrm{M}: && \alpha = -1, && \beta = 2, && \gamma = 0, && 1 < x \le 2,\\
&\mathrm{M}_{-1}: && \alpha = -1, && \beta = 2, && \gamma = 4, && 2 < x \le 3,\\
&\mathrm{R}_{-1}: && \alpha = 1, && \beta = -4, && \gamma = 4, && 3 < x \le 4.
\end{aligned} \tag{A.6}$$

The periodic condition $f^{(n)}(X_0) = X_0$ of eq. (A.5) with the given initial value $X_0 = \lambda$ gives

$$\frac{\lambda}{2} = \sum_{i=0}^{n+1} \frac{A_i}{\lambda^i}, \tag{A.7}$$

where

$$\begin{aligned}
B_i &= \alpha_0 \ldots \alpha_i,\\
A_0 &= -\tfrac{1}{2}\alpha_0\beta_0,\\
A_i &= -\tfrac{1}{2}B_{i-1}(\gamma_{i-1} + \alpha_i\beta_i), \quad i = 1, \ldots, n-2,\\
A_{n-1} &= \tfrac{1}{2}B_{n-2}(1 - \gamma_{n-2}).
\end{aligned} \tag{A.8}$$

This is called the λ-expansion of $\lambda/2$. For any given symbolic sequence of length n, the related λ-expansion can be constructed by eq. (A.6) and eq. (A.8). There is a 1–1 correspondence between the admissible sequences found numerically for periodic sequences with periods less than 6 and those computed using the λ-expansion method. Further extensions of the λ-expansion method should be possible but the algebra becomes tedious as the number of symbols increases.

Appendix B

Recursion formulae for the number of periodic cycles of circle maps

Recursion formulae for the number of periodic cycles for the $T = 0$ and 1 circle maps can be developed based on analysis of bifurcations. Since the basic methods have been developed in earlier publications, here we only give the main formulae and results. Another approach, based on modifying formulae of a classic combinatorial group studied by Fine [39] and Gilbert and Riordan [40], also gives identical results [45].

For some orbits with even periods in circle maps, the periodic points can pass through both critical points of the map symmetrically. The symbolic sequence is $X_j \mathrm{P} X_j' \overline{\mathrm{P}}$, where X_j and X_j' are two critical points of the map, P and $\bar{\mathrm{P}}$ are related to each other by changing the symbols $\mathrm{R}_j \leftrightarrow \mathrm{L}_j$ and $\mathrm{M}_j \leftrightarrow \mathrm{M}_{-j}$. When a symmetric periodic orbit loses stability, simultaneously an asymmetric periodic orbit with the same length of the cycle becomes stable. This kind of transition has been called the split bifurcation [41].

The number of periodic cycles for quadratic [37, 32] and cubic [31, 32] maps can be calculated by developing recursion formulae based on an analysis of the types of bifurcations of the maps. The same technique enables us to derive recursion formulae for circle maps.

(i) $T=0$ circle map, $\tau=0$, $0<A<1$. Let $M(d)$ represent the number of orbits of period d that arise from tangent bifurcations and $P(d)$ represent the number of orbits of period d that arise from pitchfork (period-doubling) bifurcations. We have

$$4^n = \sum\nolimits_{d/n} d\,[2M(d) + P(d)] \tag{B.1}$$

where $P(n) = P(n/2) + M(n/2)$, if n is even; $P(n) = 0$, if n is odd. The symbol $\Sigma_{d/n}$ counts all factors of n including 1 and n. The justification for this formula is as follows. If we are only concerned about periodicity, the $T=0$ circle map is equivalent to a fourth order polynomial $g(x)$. The periodic condition for a circle of length n is $g^{(n)}(x) = x$ which is a 4^n order polynomial. When the map is onto ($\tau=0$, $A=1$ in $T=0$ circle maps) the equation $g^{(n)}(x)=x$ gives 4^n real roots which must be equal to the total number of periodic points of period n that arise both from tangent and pitchfork bifurcations. Any smaller period d that divides n is also a period n orbit. Every stable periodic orbit first arises from a tangent bifurcation, always accompanied by an unstable periodic orbit. With initial conditions $M(1)=2$ and $P(1)=0$, $M(n)$ can be calculated by the recursion formula

$$M(n) = \frac{1}{2n}\Bigg(4^n - nP(n) - \sum_{\substack{d/n \\ d<n}} d\,[2M(d) + P(d)]\Bigg). \tag{B.2}$$

Results of eq. (B.2) for $n \le 11$ are given in table 1.

(ii) $T=1$ circle map, $\tau=0$, $0<A<0.73$. In this parameter region the number of periodic cycles is the same as the cubic map [31–32, 42]. The recur-

Table 1
The number of periodic cycles determined by the recursion formulae, eqs. (B.2) and (B.4). $M(n)$ is the number of periodic cycles arising from the tangent bifurcations. $P(n)$ is the number of periodic cycles arising from the period-doubling bifurcations.

Period	$T=0,\ \tau=0,\ 0<A<1$		$T=1,\ \tau=0,\ 0<A<1.73$	
n	$M(n)$	$P(n)$	$M(n)$	$P(n)$
1	2	0	3	1
2	2	2	8	4
3	10	0	56	0
4	28	4	288	12
5	102	0	1680	0
6	330	10	9744	56
7	1170	0	58824	0
8	4064	32	350000	300
9	14560	0	2241848	0
10	52326	102	14122080	1680
11	190650	0	89878488	0

sion formula for $M(n)$ is given by

$$3^n = \sum_{d/n} d\,[2M(d) + P(d)] + S(n), \qquad \text{(B.3)}$$

where $P(n) = M(n/2) + P(n/2)$ if n is even; $P(n) = 0$ if n is odd. Here $S(n)$ represents the roots of the symmetric period 2 cycle that arises from a period-doubling bifurcation and that loses stability by a split bifurcation. $S(n) = 2$ if n is even; $S(n) = 0$ if n is odd. The initial conditions are $M(1) = 1$, $P(1) = 1$.

(iii) $T = 1$ circle map. $\tau = 0$, $0 < A < 1.73$. The derivation of the recursion formula in this case is similar to the previous ones. The recursion formula for $M(n)$ is

$$7^n = \sum_{d/n} d\,[2M(d) + P(d)] + S(n), \qquad \text{(B.4)}$$

where $P(n) = P(n/2) + M(n/2)$, if n is even; $P(n) = 0$, if n is odd; $S(n) = 0$ if n is odd; $S(n) = 2$ if n is even. The initial conditions are $M(1) = 3$, $P(1) = 1$. The results of $M(n)$ for $n \le 11$ are compiled in table 1. Since the split bifurcation does not change the cycle length, in table 1 we do not account for the number of periodic orbits that arise from split bifurcations.

References

[1] G.M. Zaslavsky, Phys. Lett. A 69 (1978) 145.
[2] M.R. Guevara and L. Glass, J. Math. Biol. 14 (1983) 1.
[3] D.L. Gonzalez and O. Piro, Phys. Rev. Lett. 50 (1983) 870.
[4] J.P. Keener and L. Glass, J. Math. Biol. 21 (1984) 175.
[5] E.J. Ding, Phys. Rev. A 35 (1987) 2669.
[6] M.R. Guevara, L. Glass and A. Shrier, Science 214 (1981) 1350.
[7] L. Glass, M.R. Guevara, A. Shrier and R. Perez, Physica D 7 (1983) 89.
[8] L. Glass, M.R. Guevara, J. Bélair and A. Shrier, Phys. Rev. A 29 (1984) 1348.
[9] M. Dolnik, I. Schreiber and M. Marek, Phys. Lett. A 100 (1984) 316.
[10] K. Ikeda, H. Daido and O. Akimoto, Phys. Rev. Lett. 45 (1980) 709.
[11] F.A. Hopf, D.L. Kaplan, H.M. Gibbs and R.L. Shoemaker, Phys. Rev. A 25 (1982) 2172.
[12] P. Mandel and P. Kapral, Opt. Commun. 47 (1983) 151.
[13] M.J. Feigenbaum, L.P. Kadanoff and S.J. Shenker, Physica D 5 (1982) 370.
[14] S. Ostlund, D. Rand, J. Sethna and E. Siggia, Physica D 8 (1983) 303.
[15] T. Bohr, P. Bak and M.H. Jensen, Phys. Rev. A 30 (1984) 1970.
[16] P. Bak, T. Bohr and M.H. Jensen, in: Directions in Chaos, Vol. 2, B.L. Hao, ed. (World Scientific, Singapore, 1988).
[17] J. Guckenheimer, in: Dynamical Systems, C.I.M.E. Lectures (Birkhauser, Boston, 1980).
[18] M. Levi, Mem. Am. Math. Soc. 32 (1981) No. 244.
[19] L. Glass and R. Perez, Phys. Rev. Lett. 48 (1982) 1772.
[20] M. Schell, S. Fraser and R. Kapral, Phys. Rev. A 28 (1983) 373.
[21] J. Bélair and L. Glass, Phys. Lett. A 96 (1983) 113.
[22] J. Bélair and L. Glass, Physica D 16 (1985) 143.
[23] P. Boyland, Commun. Math. Phys. 106 (1986) 353.
[24] R.S. MacKay and C. Tresser, Physica D 19 (1986) 206.
[25] N. Metropolis, M.L. Stein and P.R. Stein, J. Comb. Theor. A 215 (1973) 25.
[26] J. Guckenheimer, Inventiones Math. 39 (1977) 165.
[27] B. Derrida, A. Gervois and Y. Pomeau, Ann. Inst. Henri Poincaré 29A (1978) 305.
[28] W.Z. Zeng, M.Z. Ding and J.N. Li, Chinese Phys. Lett. 2 (1985) 293.
[29] O. Chavoya-Aceves, F. Angulo-Brown and E. Pina, Physica D 14 (1985) 374.
[30] B.L. Hao, Physica A 140 (1986) 85.
[31] B.L. Hao and W.Z. Zeng, in: The XV International Colloquium on Group Theoretical Methods in Physics (World Scientific, Singapore, 1987).
[32] W.Z. Zeng, Ph.D. Thesis, Institute of Theoretical Physics, Peking (1987).
[33] R.S. MacKay and C. Tresser, Physica D 27 (1987) 412.
[34] S.J. Chang, M. Wortis and J. Wright, Phys. Rev. A 24 (1981) 2669.
[35] K. Hockett and P. Holmes, Proc. IEEE 75 (1987) 1071.
[36] A. Renyi, Acta Math. Acad. Sci. Hung., t.8 (1957) 477.
[37] P. Holmes and D. Whitley, Phil. Trans. R. Soc. Lond. A 311 (1984) 43.
[38] T.D. Rogers and D. Whitley, Math. Modelling 4 (1983) 9.
[39] N.J. Fine, Illinois J. Math. 2 (1958) 285.
[40] E.N. Gilbert and J. Riordan, Illinois J. Math. 5 (1961) 657.
[41] J. Testa and G.A. Held, Phys. Rev. 28 (1983) 3085.
[42] W.Z. Zeng, Commun. Theor. Phys. 8 (1987) 273.
[43] G.H. Gunaratne, M.H. Jensen and I. Procaccia, Nonlinearity 1 (1988) 157.
[44] J.A. Glazier and A. Libchaber, IEEE Trans. Circuits Syst. 35 (1988) 790.
[45] W.-Z. Zeng, unpublished results.