SELF-SIMILARITY IN PERIODICALLY FORCED OSCILLATORS

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Periodic stimulation of limit cycle oscillations leads to one-dimensional maps $f: S^1 \to S^1$ with two parameters which correspond to the frequency and amplitude of the periodic forcing. Bifurcations are described for the situtation in which f is of topological degree zero. Self-similarity is found in the parameter space.

Periodically forced nonlinear oscillators display a rich spectrum of dynamics, including bistability, hysteresis, period-doubling bifurcations, intermittency and aperiodic "chaotic" dynamics [1–7]. No clear understanding of the global bifurcation structure as frequency and amplitude of the forcing are varied has yet been obtained. However, in many instances the dynamics can be approximated by one-dimensional maps [8–18]. For the special case in which one stimulates a strongly attracting limit cycle oscillation by brief stimuli, the reduction to a one-dimensional map is straightforward [2,8,9,13,17,18]. In this case, one obtains the one-dimensional finite difference equation

$$x_{i+1} = g(x_i) + \tau = f(x_i) \quad (\text{mod } 1)\,. \tag{1}$$

Here x_i represents the phase in the cycle (which is normalized to lie between 0 and 1) at which the ith stimulus falls, τ is the time interval between successive stimuli, and g is a nonlinear function $g: S^1 \to S^1$. Experiments in a preparation of spontaneously beating cardiac cells from chick heart perturbed by brief electrical current pulses displayed period-doubling bifurcations and irregular aperiodic dynamics [2]. There was close agreement with predictions made by iterating (1) using an experimentally determined $g(x)$.

The topological degree of $g(x)$ [hence of $f(x)$] counts the number of times g traverses the unit circle as x increases around the unit circle once. For example the function

$$g(x) = Mx + b\sin(2\pi x)\,, \tag{2}$$

where M is an integer, has a topological degree of M. Many biological oscillators show experimentally measured functions $g(x)$ with a topological degree of 1 for small-amplitude stimuli and topological degree of 0 for large-amplitude stimuli [14,19]. The particular choice of (2) with $M = 1$ arises in many different applications and has been studied in some detail [9,13,15–17,20–24]. In this note we consider the bifurcations of (1) for maps of degree zero. Only limited results have so far been obtained for maps of degree zero [14,18].

Repeated iteration of (1) generates a sequence of points $x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), \ldots$. If

$$x_{i+N} = x_i; \quad x_{i+j} \neq x_i\,, \quad 1 \leqslant j < N\,, \tag{3}$$

then the sequence $x_0, x_1, \ldots, x_N = x_0$ is a cycle of period N which is stable provided $|(\partial f^n/\partial x_i)_{x_i = x_0}| < 1$. If an extremum of $f(x)$ is a point on the cycle then the cycle is called superstable. The locus of superstable cycles in parameter space is called the skeleton [16,17]. Since $g(x)$ is considered mod 1 in (1), the skeleton is periodic in τ with a period of 1.

We now consider the skeleton of (1) when $g(x)$ is of topological degree zero. Rather than consider the sine functions, $g(x) = b\sin(2\pi x)$ (mod 1), we let

$$\begin{aligned} g(x) &= 8bx(1-2x)\,, & 0 \leqslant x \leqslant 0.5\,, \\ &= 8b[x(2x-3)+1]+1\,, & 0.5 \leqslant x \leqslant 1\,. \end{aligned} \tag{4}$$

The computations we give here for (4) have been repeated for (2) (with $M = 0$) and give qualitatively similar results.

The branches of the skeleton for cycles of period 1 can be simply computed. The local maximum ($x = 0.25$) is on a cycle of period 1 on the line $b = 0.25 - \tau + j$ and the local minimum ($x = 0.75$) is on a cycle of period 1 on the line $b = 0.25 + \tau + j$ where j is an integer. Since no two branches of the skeleton associated with the maximum can intersect, the slope of all such branches as b increases approaches -1. Likewise, the slope of all branches of the skeleton associated with the minimum approaches +1.

Fig. 1a shows the skeleton up to period 3. Branches of the skeleton associated with the maximum are designated by a + sign and those associated with the minimum by a − sign. On the line $\tau = 0$, with $0 \leqslant b \leqslant 0.5$, (4) is two disjoint quadratic functions. Thus, on the line $\tau = 0$ and $0 \leqslant b \leqslant 0.5$ the same bifurcations are observed as in interval maps with one maximum [25,26]. For the region $0 \leqslant \tau + b \leqslant 0.5$, the interval (0, 0.5) is invariant and the branches of the skeleton which are present at $\tau = 0$ extend smoothly into this region. The behavior in this region is thus a straight-

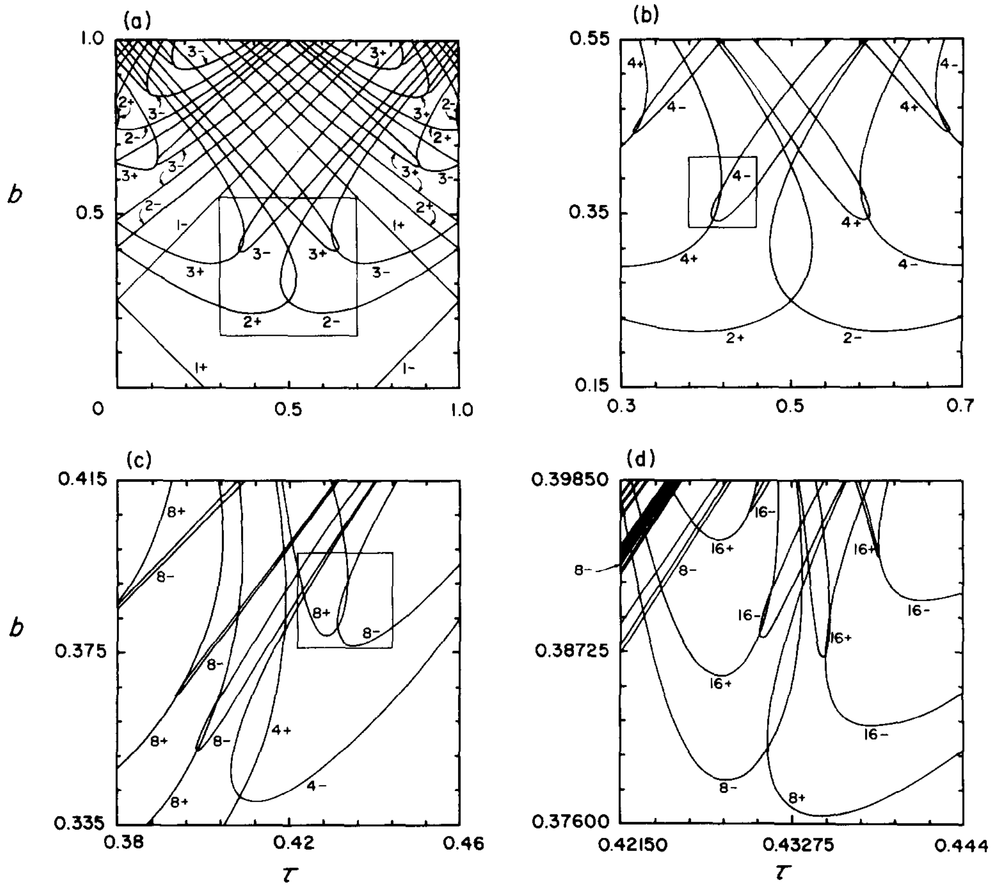

Fig. 1. The skeleton for a degree zero circle map generated by substituting (4) in (1); $k+$ represents a cycle of period k associated with the maximum and $k-$ represents a cycle of period k associated with the minimum. (a) Locus of superstable cycles of periods 1, 2, 3. To facilitate drafting, not all cycles of period 3 are labelled. (b) Enlargement of the square in (a); the superstable branches of periods 2 and 4 are now shown. (c) Enlargement of the square in (b); periods 4 and 8 are represented. Notice the topologic equivalence with (b). (d) Enlargement of the square in (c); periods 8 and 16 only are shown. In the upper left corner are 2 lines representing period 8 as labelled. The remaining lines in the upper left-hand corner are all of period 16 (there are 16 in all but these are not resolved in this figure).

forward extension of the well-known results about interval maps with one maximum. Thus, for fixed τ as b increases periodic orbits appear in the U-sequence [26].

Along the line

$$b + \tau = 0.75 \quad (5)$$

the first iterate of the maximum is the minimum. Consequently, along the line in (5) branches of the skeleton associated with the maximum and minimum must intersect. This creates a whole new sequence of skeletal branches associated with the minimum which are also ordered according to the U-sequence.

We have done numerical studies of the bifurcation structure in skeletal branches tangent to the line in (5). In fig. 1b we show the period 2 and period 4 cycles in the enclosed square in fig. 1a. Figs. 1c and 1d show magnifications of regions in parameter space focusing on period 4 and 8 cycles, and period 8 and 16 cycles, respectively. The apparent repetition of the same geometric features in figs. 1b–1d on different length scales is called self-similarity. Self-similarity can also be found for other period-doubling sequences (e.g. 3, 6, 12 ...).

Self-similarity in the bifurcations of two-parameter quartic maps was observed by Chang et al. [27]. Self-similarity has also been observed in bifurcations leading to intermittency in circle maps of degree 1 [24]. As well, in the final stages of preparation of this manuscript we became aware of an independent study of self-similarity of period-doubling bifurcations for circle maps of degree 1 which gives results related to those shown in fig. 1 [28]. Thus, self-similar bifurcations in parameter space may well be a fundamental feature of bifurcations in two-parameter maps. Note that in the one-parameter interval maps there is already self-similarity of the bifurcation diagram in parameter space, since the ratio of the sizes of successive period-doubling zones converges to Feigenbaum's number [25]. However, the rich geometrical structure present in two-parameter maps shown in fig. 1 is not present in the one-parameter examples.

Previous studies have found the skeleton shown in fig. 1a in two different situations: (i) in the bifurcations of two-parameter cubic maps [29]; and (ii) in the bifurcations of circle maps of degree 1 [16,17,28]. In the skeleton of degree 1 maps it was conjectured that the skeleton in fig. 1 appeared an infinite number of times but in an orderly fashion as described in refs. [16,17]. We believe that this bifurcation structure represents the unfolding of bifurcations in two-parameter, one-dimensional maps with two extrema, and thus may be widely observed. Numerical results on a two-dimensional, two-parameter map indicate that similar structures may also be present [30].

As one moves about in the two-parameter space of fig. 1 one will observe many features observed from periodically forced oscillations including bistability, hysteresis, period-doubling bifurcations, intermittency and chaos. The great complexity but orderly structure of fig. 1 makes it clear that experimentalists should try, if at all possible, to study bifurcations as a function of two parameters. Otherwise, it will be difficult to understand the sequence of bifurcations experimentally observed.

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