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Ordered and disordered dynamics in random networks

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Abstract. – Random Boolean networks that model genetic networks show transitions between ordered and disordered dynamics as a function of the number of inputs per element, K, and the probability, p, that the truth table for a given element will have a bias for being 1, in the limit as the number of elements $N \to \infty$. We analyze transitions between ordered and disordered dynamics in randomly constructed ordinary differential equation analogues of the random Boolean networks. These networks show a transition from order to chaos for finite N. Qualitative features of the dynamics in a given network can be predicted based on the computation of the mean dimension of the subspace admitting outflows during the integration of the equations.

Genetic networks have been modeled by random Boolean networks in which time is discrete and each element computes a Boolean function based on the values of inputs to that element [1]. Since the number of human genes is of the order of 100,000, and each gene is idealized as either on (1) or off (0), the state space for the human gene activity is huge. An order-disorder transition has been described for random Boolean networks in the limit that the number of variables, $N \to \infty$, as a function of the number of inputs per variable, K, and the probability, p, that the truth table for a given element will have a bias for being 1 [2]-[8]. The order-disorder boundary is given by

$$K_{\mathrm{c}} = \frac{1}{2p_{\mathrm{c}}(1-p_{\mathrm{c}})}, \tag{1}$$

where K_{c} and p_{c} represent the values of K and p on the boundary [2], [4], [6]. Kauffman has argued that for a network to be biologically meaningful, it should have relatively few attractors,

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and the cycle length of attractors should be comparatively short [1]. In real biological systems there are not clocking devices to generate synchronous updating and theoretical models are more appropriately formulated as continuous differential equations [9]-[14]. Here we present numerical evidence for an order-disorder transition in differential equation analogues of the discrete switching networks. We also present a probabilistic model of the dynamics, in which we show that (1) also applies to the continuous equations.

First consider a logical network consisting of N binary variables, $X_i = 0, 1,\ \ i = 1, \dots, N$. Since we consider the networks as models of genes, X_i represents the activity of gene i. In other contexts, logical variables may represents spins or voltages. The network is updated by means of the dynamical equation

$$X_i(j+1) = \Lambda_i(X_{i_1}(j), X_{i_2}(j), \dots, X_{i_K}(j)), \qquad i = 1, \dots, N, \tag{2}$$

where $\Lambda_i(X_{i_1}(j), X_{i_2}(j), \dots, X_{i_K}(j)) \in \{0, 1\}$ and K is the number of inputs. More compactly, we have

$$\mathbf{X}(j+1) = \Lambda(\mathbf{X}(j)), \qquad i = 1, \dots, N\,. \tag{3}$$

Thus, for any state $\mathbf{X}(j)$, Λ is a truth table determining $\mathbf{X}(j+1)$.

The logical structure of eq. (2) can be captured by a differential equation [9]. To a continuous variable $x_i(t)$, we associate a discrete variable $X_i(t)$,

$$X_i(t) = 0 \ \text{ if } \ x_i(t) < 0; \ \text{ otherwise } \ X_i(t) = 1. \tag{4}$$

For any logical network, we define an analogous differential equation,

$$\frac{\mathrm{d}x_i}{\mathrm{d}t} = -x_i + \lambda_i(X_{i_1}(j), X_{i_2}(j), \dots, X_{i_K}(j)), \qquad i = 1, \dots, N, \tag{5}$$

where $\lambda_i(X_{i_1}(j), X_{i_2}(j), \dots, X_{i_K}(j))$ is a scalar whose sign is negative (positive) if the corresponding logical variable $\Lambda_i(X_{i1}(j), X_{i2}(j), \dots, X_{i_K}(j))$ is 0 (1).

For each variable, the temporal evolution is governed by a first-order piecewise linear differential equation. Let $\{t_1, t_2, \dots, t_k\}$ denote the *switch times* when any variable of the network crosses 0. The solution of eq. (5) for each variable x_i for $t_j < t < t_{j+1}$ is

$$x_i(t) = x_i(t_j)\, e^{-(t-t_j)} + \lambda_i(X_{i_1}(j), X_{i_2}(j), \dots, X_{i_K}(j))(1 - e^{-(t-t_j)}). \tag{6}$$

Depending on the particular network, eq. (5) can display steady states, limit cycles, quasiperiodicity or chaos [9]-[13]. Although the origin of chaos in one particular four-dimensional network has been analyzed [13], no general methods have yet been developed to determine whether chaotic dynamics exists in any given equation without integrating it. Chaos is the usual behavior in these networks for $N = 64$, $9 \le K \le 25$, $p = 0.5$ [14].

In carrying out the computations, we have made several additional assumptions concerning the structure of the equations. i) There is no self-input or reciprocal input. This means that i cannot be an input to itself, and if i is an input to k, k is not also an input to i. These assumptions eliminate the possibility of asymptotic approach to stable steady states in the neighborhood of threshold axes $x_i = 0$ [9], [11]. If these assumptions are not made, the techniques for analysis of the dynamics based on the symbolic dynamics described below fail. ii) We set λ_i in eq. (5) to be in the range $-1\pm \mid 0.01 \mid$ or $1\pm \mid 0.01 \mid$. iii) In the truth tables we assume that for a given value of p, half the variables of the network have entries that are biased towards 1 and half are biased towards 0.

Although the piecewise-linear nature of eq. (5) facilitates the speed and accuracy of numerical integration over other integration techniques, numerical studies nevertheless grow

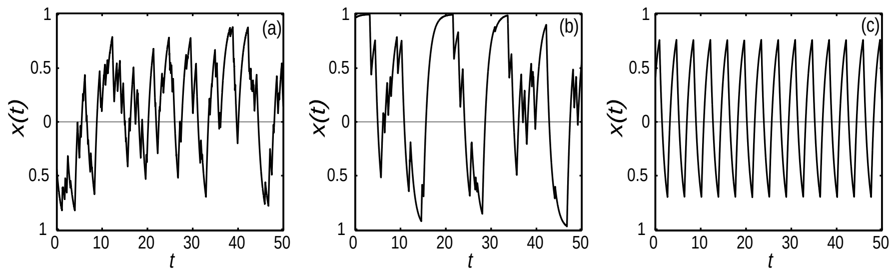

Fig. 1. – Time series for a typical variable in a network with $N = 50$, $K = 8$. (a) $p = 0.5$, (b) $p = 0.8$, (c) $p = 0.85$. In panels (a) and (b), the time series satisfy our tests for chaos and in (c) the dynamics are periodic.

large rapidly. Consequently, we focus on the order-disorder transition for limited regions of parameter space. For each network, we investigated one initial condition and classified the dynamics after a transient of 120 000 switch times. The dynamics can be classified by using symbolic dynamics, keeping track of the variable that switches as a function of time. Provided the network does not reach a steady state during the course of the integration, we generate a sequence of integers denoting the label of the variable that switches at each consecutive switch time. We search for periodicities in this sequence of integers, searching for periodicities up to length 3000.

Given the restrictions on the equations mentioned above, if the symbolic sequence is periodic, in the differential equation there is a stable limit cycle oscillation [9]. If the sequence is not periodic, then in the differential equation, there can either be quasiperiodic dynamics or chaotic dynamics. To identify quasiperiodic dynamics, we need to keep track of the exact switching times of all the variables of the network. However, pilot investigations of several hundred randomly constructed networks failed to identify quasiperiodic dynamics. Consequently, to simplify the analysis we lump quasiperiodic rhythms together with chaotic networks, where we expect that the incidence of quasiperiodic dynamics is negligible (less than 1%).

Figure 1 illustrates typical dynamics for a single variable in a network with $N = 50$, $K = 8$. The same connection matrix was assumed for three values, $p = 0.5$, 0.8, 0.85. In fig. 1(a) and 1(b), the symbolic transition sequences corresponding to the time series do not show periodicities. The Lyapunov number, which can be evaluated numerically using techniques in Ott [14], [15], is positive for panels (a) and (b) and negative for panel (c). Based on the above, the dynamics are deterministic chaos in panels (a) and (b) and periodic in panel (c).

Figure 2 shows the numbers of networks displaying steady states, limit cycles, and chaos for $K = 8$ and $N = 50$ panel (a), $N = 100$ panel (b), and $N = 200$ panel (c), for 50 different randomly generated networks for $p \in [0.5, 1.0]$ incremented in steps of 0.02. As p increases the number of chaotic networks decreases and the number of networks displaying steady states increases. The values of p associated with limit cycles are centered in an increasingly narrow range as N increases. Since the total number of orthants of phase space that are visited during the simulations is a tiny fraction of the 2^N orthants of phase space, we cannot exclude the possibility that networks classified as chaotic will eventually reach a stable steady state or limit cycle if the integration times are extended.

We now characterize the dynamics in eq. (5). Let j designate the time interval (t_j, t_{j+1}), in eq. (6). At any given time in the interval j, the state in the continuous equation is mapped to the logical state $\mathbf{X}(j)$. The distance between two logical states is the number of variables in which the activities differ. We call the distance between $\mathbf{X}(j)$ and $\mathbf{X}(j + 1)$, determined from eq. (2), the *outflow dimension*, $h(j)$, of $\mathbf{X}(j)$. The outflow dimension is a measure of the number of different variables that have the potential to cross 0 at the next switch time of the network [9]. If $h(j) = 0$, then the system will approach a steady state.

Using methods similar to those developed in earlier work by Derrida and others [2], [5], [8], we compute the mean value of h. Denote the number of variables in logical state α, and with truth table entry β in time interval j by $N_{\alpha\beta}(j)$, where $\alpha, \beta \in \{0, 1\}$. If $N_{01}(j) = N_{10}(j) = 0$, there is a fixed point. If this is not the case there will eventually be a transition to a new state. The probability that there will be a transition of a variable from state 0 to state 1 is $P_{0\to1} = N_{01}(j)/(N_{01}(j) + N_{10}(j))$, and the probability that there will be a transition of a variable from state 1 to state 0 is $P_{1\to0} = N_{10}(j)/(N_{01}(j) + N_{10}(j))$.

The expected number of inputs from the switching variable to variables in state (α, β) is $\rho N_{\alpha\beta}$, where $\rho = K/N$. Now assume that there is a transition of a variable from $0 \to 1$ at switch time t_{j+1}. We adopt a probabilistic approach to determine $N_{\alpha\beta}(j+1)$. Consider first the value of $N_{00}(j+1)$. This may be different from $N_{00}(j)$ if inputs from the variable that changed its state have inputs to variables in N_{00} that leads to a change to N_{01}, or inputs to variables in N_{01} that leads to changes to state N_{00}. Thus, we find that

$$N_{00}(j+1) = N_{00}(j) - \rho(1-p)N_{00}(j) + \rho p N_{01}(j).$$

If there is a transition of a variable from $1 \to 0$ at time t_j, then the above expression would be changed to

$$N_{00}(j+1) = N_{00}(j) + 1 - \rho(1-p)N_{00}(j) + \rho p N_{01}(j).$$

In similar fashion, by weighting the respective transition probabilities, we are led to the following system of equations for $N_{\alpha\beta}(j+1)$:

$$\begin{cases} N_{00}(j+1) = N_{00}(j) + P_{1\to0} - \rho(1-p)N_{00}(j) + \rho p N_{01}(j)\,, \\ N_{01}(j+1) = N_{01}(j) - P_{0\to1} + \rho(1-p)N_{00}(j) - \rho p N_{01}(j)\,, \\ N_{10}(j+1) = N_{10}(j) - P_{1\to0} - \rho(1-p)N_{10}(j) + \rho p N_{11}(j)\,, \\ N_{11}(j+1) = N_{11}(j) + P_{0\to1} + \rho(1-p)N_{10}(j) - \rho p N_{11}(j)\,. \end{cases} \tag{7}$$

At steady state, we have $N_{\alpha\beta}(j+1) = N_{\alpha\beta}(j) = N^*_{\alpha\beta}$, where the asterisk represents the steady-state value. Substituting this relation in eq. (7), we find that $N^*_{01} = N^*_{10}$. Substituting

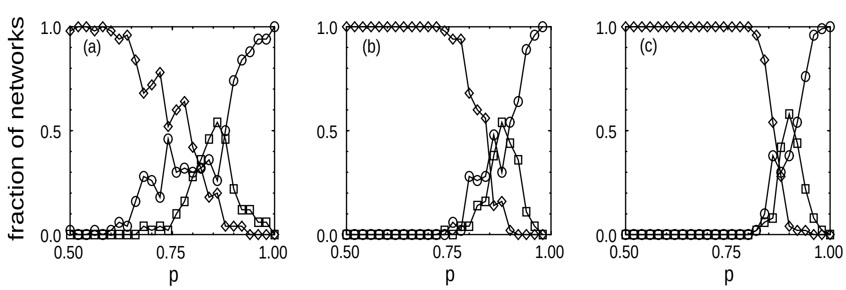

Fig. 2. – Variation in the number of networks displaying steady states (circles), limit cycles (squares), and chaos (diamonds) for $K = 8$, as a function of p, (a) $N = 50$, (b) $N = 100$, (c) $N = 200$. A single initial condition was selected for each of 50 different networks for each value of p.

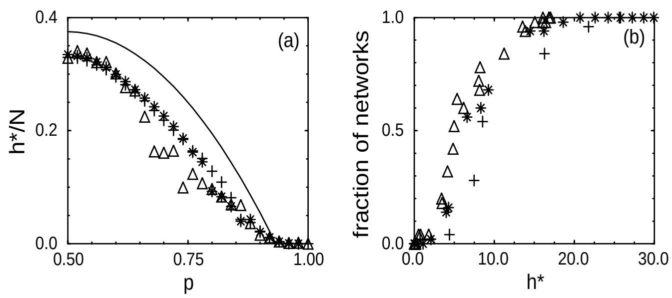

Fig. 3. – (a) Normalized mean outflow dimension, h^*/N, as a function of p for $N = 50$ (triangles), $N = 100$ (asterisks), $N = 200$ (plus signs) compared with the theoretical result from eq. (9). For each network, the value of h was averaged over 3000 switch times following a transient of 120 000 switch times. (b) Fraction of chaotic networks as a function of h^* for $N = 50$ (triangles), $N = 100$ (asterisks), $N = 200$ (plus signs).

this result in eq. (7), we find

$$\begin{cases} -\dfrac{1}{2} = -\rho(1-p)N^*_{00} + \rho p N^*_{10}, \\ -\dfrac{1}{2} = \rho(1-p)N^*_{10} - \rho p N^*_{11}. \end{cases} \tag{8}$$

Using the conservation condition, $N^*_{00}+2N^*_{10}+N^*_{11} = N$, we have three simultaneous equations. Solving these equations and recalling that at steady state the mean outflow dimension, h^*, is given by $N^*_{10} + N^*_{01} = 2N^*_{10}$, we compute

$$\frac{h^*}{N} = \frac{1}{K}(-1 + 2Kp - 2Kp^2). \tag{9}$$

In fig. 3(a), we compare the theoretical estimates from eq. (9) (solid curve) with the numerical computations for $N = 50$, 100, 200. The theoretical estimate lies consistently above the numerically computed values, but shows a similar functional dependence on p. When $h^* = 0$, we recover eq. (1). In fig. 3(b), we plot the fraction of chaotic networks as a function of h^*. The transition occurs approximately in the range $0 < h^* < 20$ for all values of N considered. The results in fig. 3 provide a challenge for further theoretical analysis.

These results have connections with the extensive studies carried out on the discrete time and discrete state space switching networks [2]-[8]. In contrast to the earlier work, in which all finite networks must eventually cycle in the limit $t \to \infty$ and which do not therefore admit deterministic chaos, in eq. (5) deterministic chaos is possible [13], [14]. At the moment, there are no general techniques to assert deterministic chaos in any given network, and it is possible that eventually networks identified as chaotic here will reach limit cycles or steady states. Nevertheless, the extremely long transient behavior would appear to render these architectures improbable for the highly constrained dynamics in real biological systems. In the continuous equations, the critical line (1) defines the line at which $h^* = 0$ and almost all networks with $p \geq p_{\mathrm{c}}$ display steady states. Thus, the continuous equations show transitions in dynamics even for finite N. As the value of p is varied from 0.5, there is a transition from chaotic dynamics to steady states, with an intervening zone of periodic dynamics that becomes increasingly narrow as N increases.

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