LEON GLASS

Looking at Dots

*The "Prof" at the Department of Machine Intelligence and Perception at the University of Edinburgh, H. C. Longuet-Higgins, had just returned from a trip to the States where he had learned of a fascinating experiment carried out by the physicist Erich Harth. The year was 1968, and I had just completed a doctorate studying the statis*tical mechanics of liquids, trying to apply my craft to the study of the brain. At the time, I did not realize that the experiment would have strong impact on the rest of my career.

The experiment was so simple that even a theoretician could do it. Take a blank piece of paper. Place this on a photocopy machine and make a copy of it. Now make a copy of the copy. This procedure is then iterated, always making a copy of the most recent copy.

Although the naïve guess might be that all copies would be blank, this was not at all the case. Small imperfections in the paper and dust on the optics of the Xerox machine introduced "noise" that arose initially as tiny specks. As the process was iterated, these tiny specks grew up—they got bigger. They did not grow to be very big, but just achieved the size of a small dot, Figure 1. The reason for this is that the optics of the photocopy machine led to a slight blurring of each dot, so that each dot grew. On the other hand, local inhibitory fields introduced by the charge transfer underlying the Xerography process limited the growth. These local fields also inhibited the initiation of new dots near an already existing dot; so that after a while (about 15 iterations), there was a pretty stable pattern of dots.

This analogue system mimicked lateral inhibitory fields that play a role in developmental biology and visual perception, and I thought it would be a fine idea to study the spatial pattern of the dots. To do this, I decided to make a transparency of the dot patterns so that I could project the dot patterns on a target pattern of concentric circles. By placing one dot at the center of the target pattern, I could count the number of dots lying in annuli a given distance away, this would give me an estimate of the spatial autocorrelation function of the dots.

But when I did this, I made a surprising finding. Superimposing the transparency of the dots upon the photocopy of the dots with a slight rotation, one obtained an image with an appearance of concentric circles (Figure 2). I described this effect and proposed a way that the visual system could process the images [1].

In 1982, David Marr called these images Glass patterns in his classic text in visual perception [2]. The effect is now well-known among visual scientists, who continue to unravel the visual mechanisms underlying the perception of these images. But despite the underlying mathematical structure of these images and the potential utility of this effect to teach mathematics, the effect is not known at all by mathematicians, as witnessed by an early rediscovery of the effect [3] and also by the description of the effect in the Spring 2000 *Mathematical Intelligencer* [4]. Let me try here to give a glimpse into the mathematical underpinnings, and to describe some of the recent psychological studies of the perception of these images.

Perceiving Vector Fields

Imagine a two-dimensional flow or vector field. We randomly sprinkle dots on the plane. Next we plot the loca-

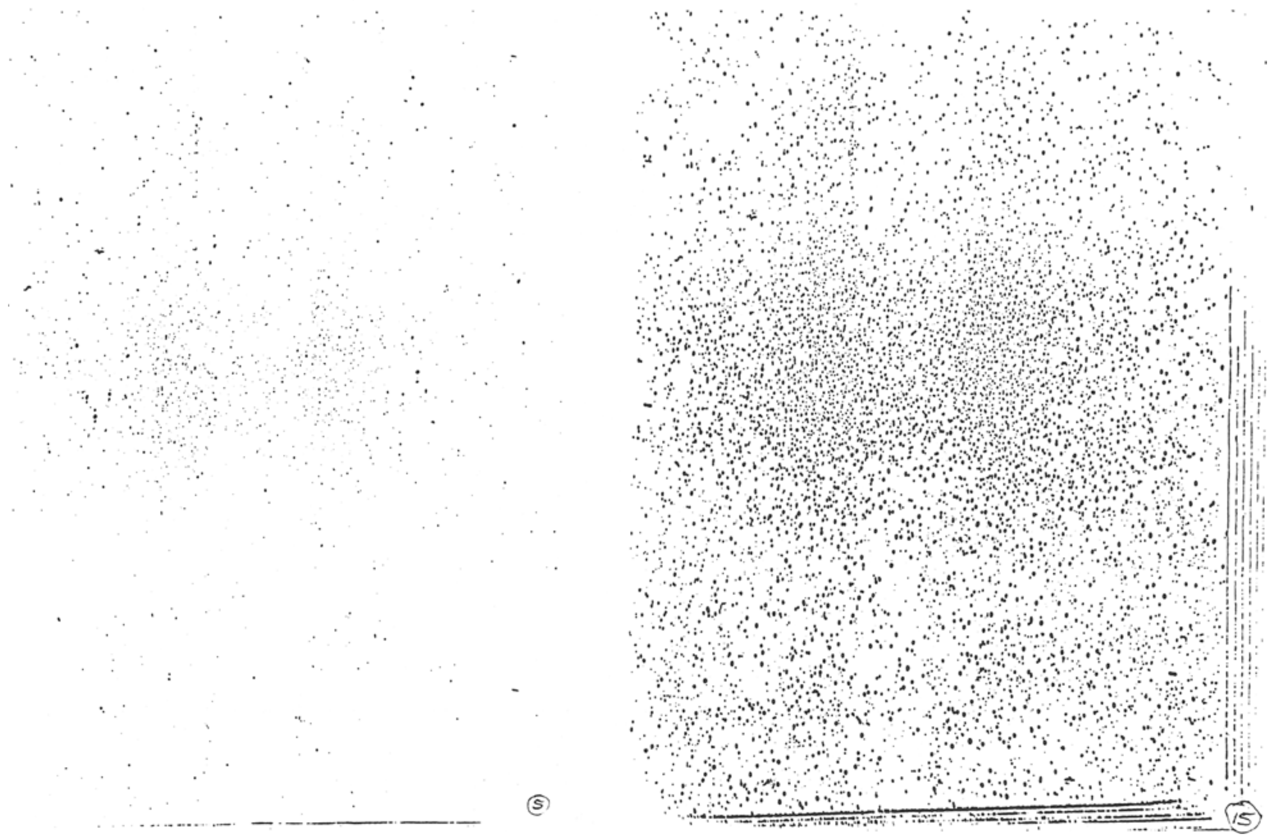


Figure 1. Original images generated in the late 1960s by making a photocopy of a blank page and then iterating the process, always taking a photocopy of the most recent copy. Images represent the output after the 5th and 15th iterations.

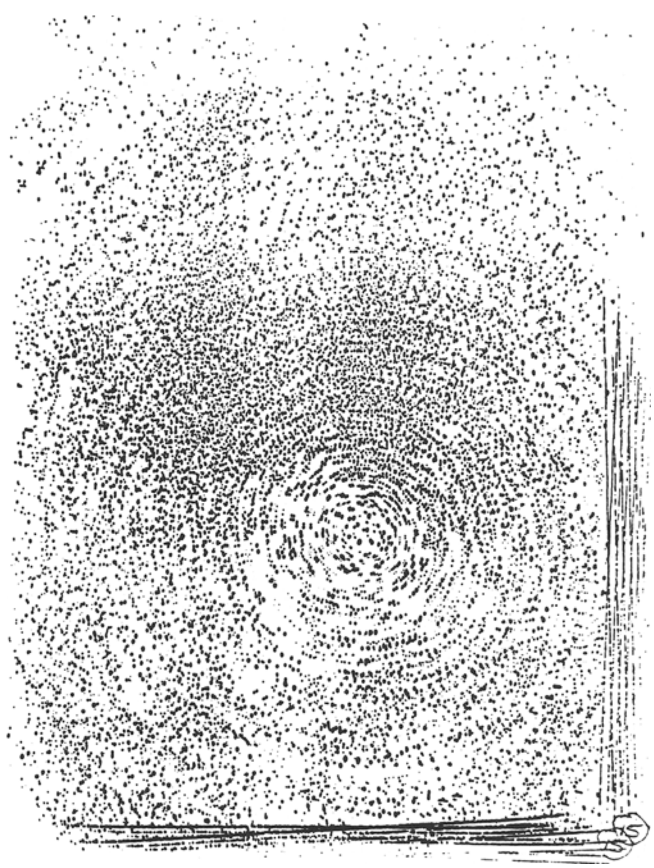


Figure 2. The image generated by superimposing a copy of the 15th iterate on itself in a rotated position.

tions of the original set of dots, and also the locations of the dots a bit later, after they have moved under the action of the flow. Provided the time interval is not too long, then when we look at the positions of both sets of dots simultaneously, we see the geometry of the vector field.

Figure 2 shows an example in which a set of dots is superimposed on itself in a rotated position to yield a circular image. But other geometries can be handled [5]. First assume that the origin is fixed, and the transformation maps each dot to a new location by a scaling of the x-coordinate by an amount a, a scaling of the y-coordinate by an amount b, and a rotation of the image about the origin by an angle θ. Then (x,y) will be transported to the position (x',y'), where

$$\begin{aligned} x' &= ax \cos \theta - by \sin \theta \\ y' &= ax \sin \theta + by \cos \theta \end{aligned} \tag{1}$$

Equation (1) is a linear map. The properties of such maps are well understood [6], [7].

What is amazing is that by looking at the images of the original set of dots combined with the superimposed dots, it is possible to perceive the underlying geometry of the transformation defined by the map. The particular geometry that results is defined by the eigenvalues of the linear transformation defined in Equation (1). The eigenvalues are the solutions of the determinant

$$\begin{vmatrix} a \cos \theta - \lambda & -b \sin \theta \\ a \sin \theta & b \cos \theta - \lambda \end{vmatrix} = 0. \tag{2}$$

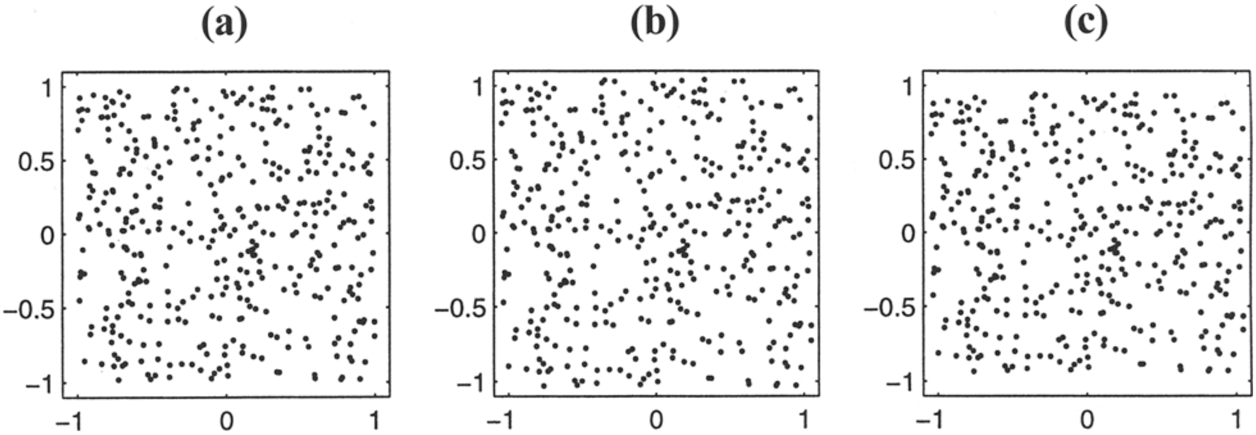


Figure 3. (a) A random pattern of 400 dots. (b) The same pattern in which the x and y coordinates are multiplied by 1.05. (c) The same pattern in which the x coordinate is multiplied by 1.05 and the y coordinate is multiplied by 0.95.

A simple computation gives those eigenvalues as

$$\lambda_{\pm} = \frac{(a+b)\cos\theta \pm \sqrt{(a-b)^2 - (a+b)^2 \sin^2\theta}}{2}. \quad (3)$$

These effects can be beautifully illustrated using transparencies of random dot patterns, and superimposing these on an overhead projector. In Figure 3a I show a random pattern of 400 dots. The x and y coordinates of each point are multiplied by 1.05 in Figure 3(b). In Figure 3(c), the x coordinates of each point are multiplied by 1.05 and the y coordinates of each point are multiplied by 0.95. The rotation of the photocopied patterns yielding circles in Figure 2 is one of the classic geometries (pure imaginary eigenvalues). Another geometry is provided by setting the center of the image as fixed, and then expanding the x and y coordinates by the same constant amount (real eigenvalues greater than 1). This yields an expanding pattern, called a "node," which is illustrated in Figure 4(a) by superimposing Figure 3(a) and 3(b). Combining expansion with rotation gives a spiral image, called a "focus," as shown in Figure 4(b), formed by superimposing Figures 3(a) and 3(b) in a rotated orientation (complex eigenvalues). Finally, if there is expansion in the x coordinate and contraction in the y coordinate, then there is a hyperbolic geometry called a "saddle" (the absolute value of one eigenvalue is greater than 1 and the absolute value of the other eigenvalue is between 0 and 1). A saddle (Figure 4c), can be generated by the superposition of Figures 3(a) and 3(c). Because these geometries can be easily appreciated without using the formulae, I always use these correlated dot images to teach about the geometry of vector fields. These geometries may even be preserved when one set of dots is one color, and the second set of dots is another color (Fig. 5a). Stan Wagon has incorporated this observation to generate colorful images of vector fields in which local flows are represented by "tear drops." The visual system integrates the local tear drop flows to give a good representation of the geometry of the vector field [8] (Fig. 5b).

Neurophysiology of Perception

Manipulation of visual images, combined with measurement of perception, or recording of electrical activity of nerve cells in the brain, provides powerful techniques to probe the functioning of the visual system. Because of the simple structure of the random dot images, visual scientists have often used them as a starting point for their investigations. I cannot summarize the many studies that have been carried out, but I will describe a couple and invite the reader to invent new visual effects that can be a probe of visual system function.

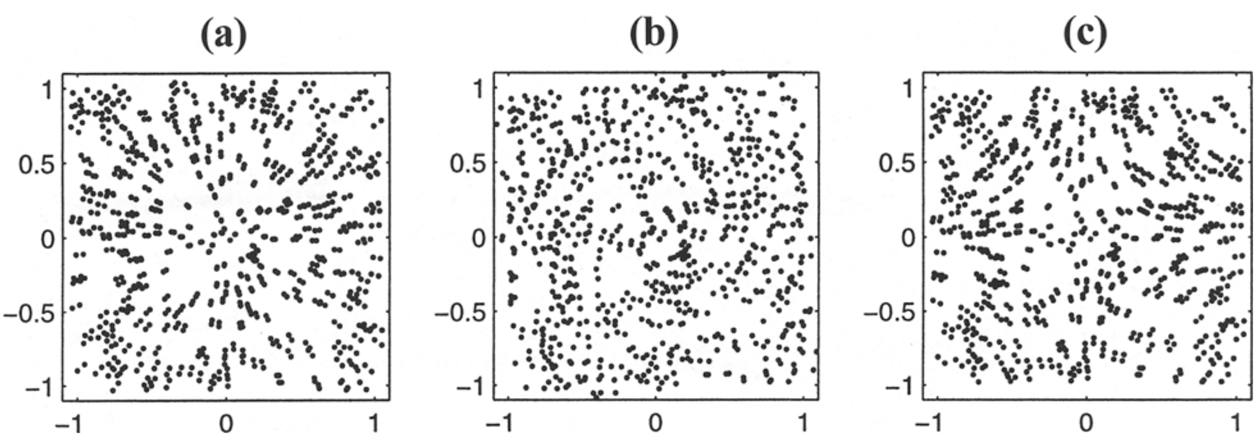
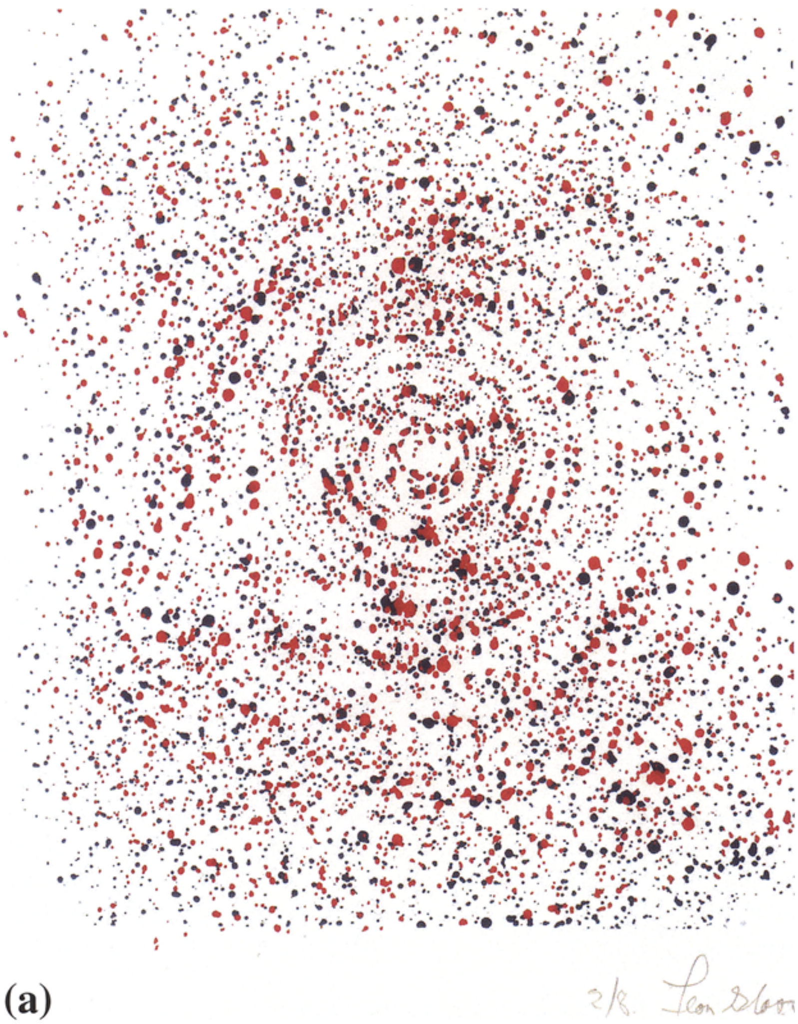
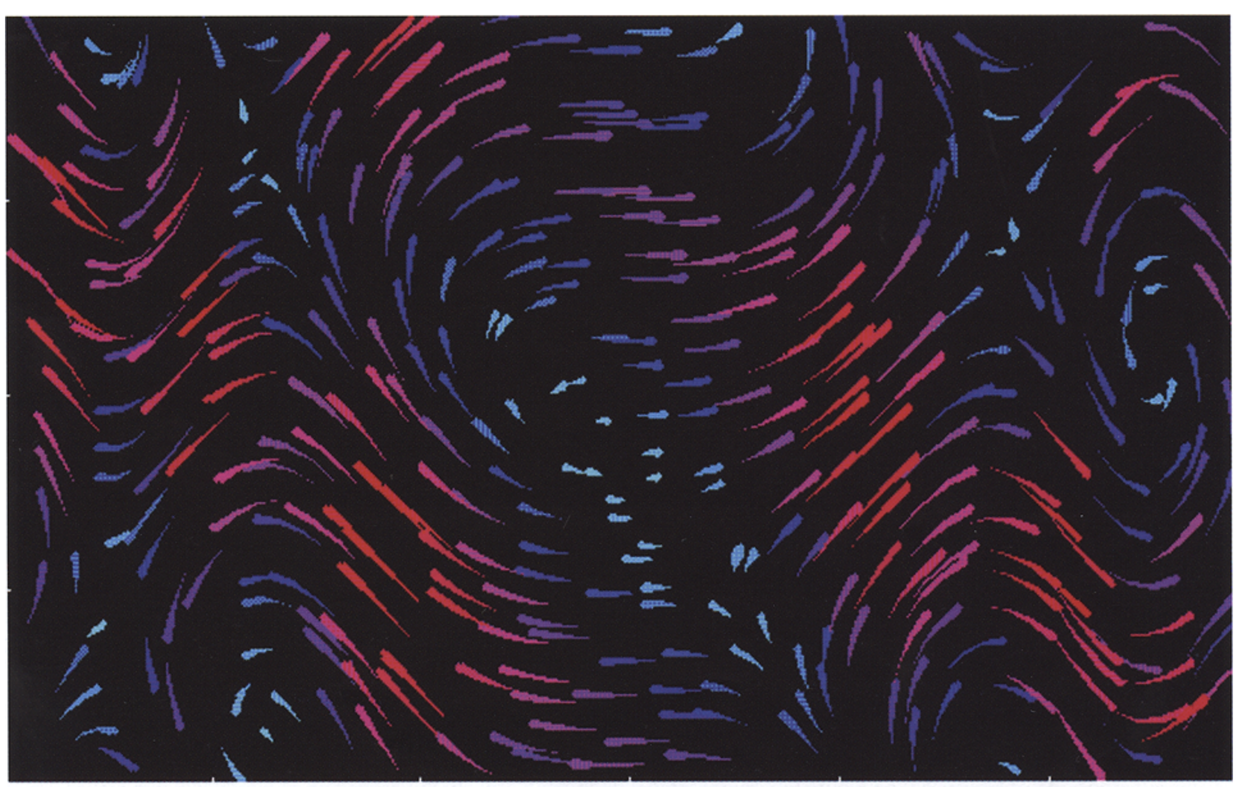


Figure 4. (a) Superposition of Figure 2(a) on Figure 2(b) to generate a node geometry. (b) Superposition of Figure 2(a) and Figure 2(b) in a rotated position to generate a focus geometry. (c) Superposition of Figure 2(a) on Figure 2(c) to generate a saddle geometry.



(a)

Figure 5 (a) A random pattern of dots generated by tossing ink on paper is superimposed on itself in a rotated position, but the two sets of dots are different colors. From a photo silkscreen print made by the author in the 1970s. (b) A colorful "tear drop" representation of vector fields from VisualDSolve (Wagon and Schwalbe [8]). Reproduced with permission from Wagon and Schwalbe [8].



(b)

In order to think about how the visual system might process the information in the dot patterns, it is useful to consider first the structure of the images. For each dot, there is a second dot that is correlated with the first dot. For example, for the circular images, the two dots always lie on the circumference of a circle centered at the point of rotation. However, in addition, there are other dots that are also in the vicinity of the first dot that lie in random directions from it. In order to detect the pattern, two steps are essential: (1) to detect the locally correlated dots and (2) to integrate the local correlations to form the global percept.

Early Nobel-Prize-winning studies of the physiology of nerve cells in the visual system of the brain carried out by Hubel and Wiesel [9] provide a basis for hypothesizing a mechanism for early stages of the detection process. Hubel and Wiesel showed that some nerve cells, called "simple cells," can be excited by lines of a particular orientation in a given region of the visual field. Consequently, two dots should also be able to stimulate a simple cell if they lie in the appropriate orientation. In a local region, there are many correlated dot pairs oriented along the flow, so cells specific for that orientation in that local region would be preferentially activated compared to cells specific for other orientations. Hubel and Wiesel also found that simple cells, specific for a certain orientation but with somewhat differing receptive fields all in the same general region of the visual field, were located in vertical columns. Further, there were cells they called "complex cells" that appeared to receive their input from simple cells lying in the same column [9]. Based on these observations, I hypothesized that the simple and complex cells in a column in the visual cortex could provide the anatomical loci to compute the local autocorrelation function of the dot patterns [1]. The integration of the outputs of the local columns to form the global percept would necessarily involve inter-columnar interactions.

Now, more than 30 years after these initial hypotheses, a large number of studies make it possible to refine and modify these ideas. Movshon and colleagues have recorded electrical activity from simple cells in the primary visual cortex (this is called area V1) of macaque monkeys while viewing dot patterns generated by superimposing a random set of dots on itself following a translation [10]. They also developed a mathematical model of the cortical cells, by assuming there were elongated excitatory and inhibitory regions of the receptive fields. A given cell would be excited (or inhibited) by dots that fell in the excitatory (or inhibitory) region of its receptive field. The good agreement between the experimentally recorded activity and the theoretical model gives support to this conceptual model of the cortical cell. Moreover, by computing the expected activity using a theoretical model, and comparing these results with the observed activity recorded experimentally, this approach is making progress in linking the separations between dots in the images presented to the monkeys with the physiological properties of individual cells.

What about the interactions between the simple cells? Zucker argues that excitatory interactions between individual cells in a given "clique" of cells, all of which have similar orientation specificity and are located in a given column, might be playing an important role in contour detection [12]. In this formulation, a "clique" of cells is carrying out the averaging operations that are necessary to compute the local autocorrelations. Thus, Zucker is hypothesizing that the columnar organization may play an important role in information processing.

This work leaves open the important question of the nature of the interactions between columns that lead to global percept. Psychophysical studies carried out by Wilson and Wilkinson pose sharp questions about the nature of the inter-columnar information processing. By partially removing some regions of the correlated dot images, they determined that the circular image, as in Figure 2, is easier to perceive than the other types of correlated dot images. Because the local information was the same in the various images, the differences in ability to perceive the images must be due to the integration steps. At the moment, it appears that these integration steps take place in a region of the brain called area V4 [11].

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Practical Implications

The random dot images may be useful in a variety of other applications. Because of the moiré effect, these images can provide a powerful method to determine a point of rotation, and to align images. Following the description of this effect in the *Scientific American*, Edward B. Seldin of Harvard Medical School developed a method to use the moiré effect to help plan maxillo-facial surgery in patients who did not have ideal alignment of the upper and lower jaws. He started out with two identical dot patterns, one fixed on the upper jaw and a second fixed on the lower jaw, initially in a superimposed orientation [14]. By manipulating images to give a better jaw alignment, it was possible to develop a plan for the surgery. More recently, Wade Schuette of Ann Arbor, Michigan demonstrated a variety of ways these effects could be used to help in alignment tasks [13].

Similar effects also arise in color printing. Colors are often represented by dots of different colors and varying sizes. Problems in alignment of the different colors can lead to undesirable moiré effects. One way to overcome these problems is for the color screens to be stochastic images. However, even when these images are stochastic, misalignment can lead to moiré effects. Such problems are being addressed by Lau [15], who recently rediscovered these phenomena in the context of commercial printing.

Do It Yourself

Some of the figures in this article are generated using the Matlab programming language. Many readers are familiar with Matlab or have access to a computer that has Matlab capabilities. The following program should be called idots.m.

```
function [e]=idots(a,b,theta)
x=ones(2,400)-2*rand(2,400);
R=[cos(theta) -sin(theta);
sin(theta) cos(theta)];
S=[a 0;
0 b];
xnew=R*S*x;
x=[x xnew];
plot(x(1,:),x(2,:),'.');
axis([-1.1 1.1 -1.1 1.1]);
axis('square')
title('Glass Pattern');
e=eig(R*S);
txt=['with eigenvalues: ' num2str(e(1)) '
  and ' num2str(e(2))];
xlabel(txt);
```

To run this program you need to open up Matlab. Figure 3(a) shows a random pattern of dots; in Figure 3(c) the x coordinate is multiplied by 1.05, the y coordinate is multiplied by 0.95, and there is no rotation. To superimpose two sets of dots with the same transformation as in Figures 3(a) and (c), type idots(0.95,1.05,0). What you obtain should look like Figure 4(c) even though the exact coordinates of the random dots will be different. (Matlab generates a different sequence of random numbers on each trial.)

Here's a problem. Superimpose two sets of random dots: an original pattern and one in which the scalings in the x and y coordinates are 1.05 and 0.95, respectively. However, vary the rotation angle. For example, in Figure 6, I show the superposition of two patterns with a rotation of about 2.61 degrees (left panel), which gives a saddle geometry, and 5.47 degrees (right panel), which gives a spiral geometry. As the angle of rotation is varied between those two values, do you ever obtain a node geometry? With Matlab, one can explore the question numerically. However, it is really better to compute the eigenvalues analytically using Eq. (3). If you do this, you will find that there is a narrow range in which you must pass through the node geometry. Here is the insight. In the spiral geometry, the two eigenvalues are complex numbers with real part less than 1, and in the hyperbolic geometry the eigenvalues are real with one eigenvalue being positive greater than 1, and the other eigenvalue positive less than 1. As θ varies, the values of the eigenvalues change continuously. Both eigenvalues first become real and less than 1, before both eigenvalues become complex. This is a great way to illustrate bifurcations in dynamical systems.

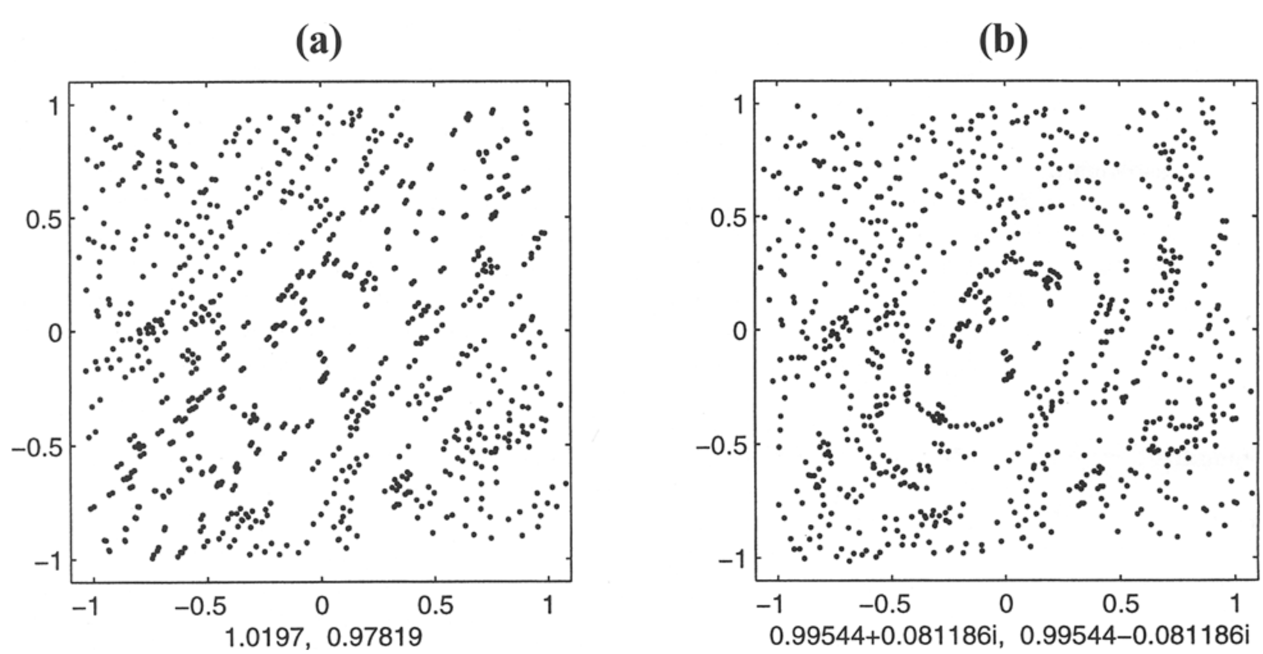

Figure 6. (a) Saddle geometry generated from two sets of correlated random dots by using the Matlab program with $a = 1.05$, $b = 0.95$, $\theta \approx 2.61°$. (b) Focus geometry generated with $a = 1.05$, $b = 0.95$, $\theta \approx 5.47°$. For $a = 1.05$, $b = 0.95$, is there a value of θ in the range $2.61° < \theta < 5.47°$ that gives a node geometry?

Now more than 30 years after I first observed these images composed of correlated random dots, it still seems we are just at the beginning of developing an understanding of how the visual system processes the information contained in these images. These images combine both local and global features, which can be varied independently. Observation of experimental subjects (men, monkeys, or even pigeons! [16]) looking at the dot patterns is providing a window into the physiological processes of vision.

REFERENCES

[1] L. Glass, Moiré effect from random dots, *Nature* **223**, 578–580 (1969).

[2] D. Marr, *Vision*, Freeman, San Francisco, 1982.

[3] J. Walker, The amateur scientist, *Scientific American* **242** (April 1980).

[4] Why circles?, *Mathematical Intelligencer* **22**, no. 2, 18 (2000).

[5] L. Glass, R. Perez. Perception of random dot interference patterns. *Nature* **246**, 360–362 (1973).

[6] R. L. Devaney, *A First Course in Chaotic Dynamical Systems*. Perseus (1992).

[7] S. Strogatz, *Nonlinear Dynamics and Chaos: With Applications to Physics, Biology, Chemistry and Engineering*, Perseus (1994).

[8] S. Wagon and D. Schwalbe, *VisualDSolve: Visualizing Differential Equations with Mathematica*, Springer/TELOS (1997). However, in some cases, dots of different colors can NOT be used to capture the geometries of vector fields. L. Glass and E. Switkes, Pattern recognition in humans: correlations which cannot be perceived. Perception **5**, 67–72 (1976).

[9] D. H. Hubel, T. N. Wiesel, Receptive fields, binocular interaction and functional architecture in the cat's visual cortex. *J. Physiol. (Lond.)* **160**, 106–154 (1962).

[10] M. A. Smith, W. Bair, and J. A. Movshon, Signals in macaque V1 neurons that support the perception of Glass patterns, *Journal of Neuroscience*, In Press (2002).

[11] H. R. Wilson and F. Wilkinson. Detection of global structure in Glass patterns: implications for form vision, *Vision Research* **38**, 2933–2947 (1998).

[12] S. W. Zucker, Which computation runs in visual cortical columns? In: *Problems in Systems Neuroscience*, J. L. van Hemmen and T. J. Sejnowski (eds.) Oxford University Press, in press (2002).

[13] W. Schuette, Glass patterns in image alignment and analysis. United States Patent 5,613,013.

[14] J. Walker, The amateur scientist, *Scientific American* **243** (November 1980).

[15] D. L. Lau, A. M. Kahn, G. R. Arce. Minimizing stochastic moiré in FM halftones by means of green-noise masks. *Journal of the Optical Society of America*. **19**, no. 11 Nov (2002).

[16] D. M. Kelly, W. F. Bischof, D. R. Wong-Wylie, et al. Detection of Glass patterns by pigeons and humans: Implications for differences in higher-level processing, *Psychol. Sci.* **12**, 338–342 (2001).

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The portrait here is a daguerrotype by Robert Shlaer; used by permission.