BISTABILITY, PERIOD DOUBLING BIFURCATIONS AND CHAOS IN A PERIODICALLY FORCED OSCILLATOR

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A two parameter mathematical model for a periodically forced nonlinear oscillator is analyzed using analytical and numerical techniques. The model displays phase locking, quasiperiodic dynamics, bistability, period-doubling bifurcations and chaotic dynamics. The regions in which the different dynamical behaviors occur as a function of the two parameters is considered.

Recent experimental studies [1,2] of periodically forced nonlinear oscillators have shown the presence of period-doubling bifurcations and chaotic dynamics obeying universal scaling properties predicted by simple theoretical models [3–5]. Although a detailed theoretical analysis of [1,2] has not yet appeared it has been shown that the dynamics of a periodically forced nonlinear oscillator can often be represented by a one dimensional Poincaré map, f, where f: $S^1 \to S^1$ [6–11]. For example, such a reduction is possible for periodically forced relaxation oscillations [7–9], the periodically driven Josephson junction [11], and periodic stimulation of spontaneously active cardiac cells [10].

In the following we consider the map

$$x_{t+1} = f(a, b, x_t) = x_t + a + b \sin 2\pi x_t \quad (\mathrm{mod}\ 1)\ , \quad (1)$$

where a, b are positive real numbers. For $b < 1/2\pi$, f is a diffeomorphism of the circle and the dynamics are well understood [12–15]. For $b = 1/2\pi$ novel scaling behavior has recently been observed [16]. For $b > 1/2\pi$, although chaotic dynamics have been observed [6,11], a detailed analysis of the dynamics as a function of a and b is not available. In the following, we mainly consider the dynamics of (1) for $b > 1/2\pi$.

The map in (1) can arise from the following process. Assume there is a variable $y(t)$ and a threshold, $\theta(t)$,

$$\theta(t) = 1 + \beta \sin 2\pi t\ , \quad (2)$$

where $0 \leqslant \beta < 1$. The variable $y(t)$ is a piecewise linear function of time. The slope of $y(t)$ is $-\alpha$, where α is a positive real number, everywhere except at isolated times $t_0, t_1, \ldots t_j$. At t_j the variable discontinuously jumps from zero to $\theta(t_j)$ (fig. 1). Thus,

$$t_{i+1} = t_i + 1/\alpha + (\beta/\alpha) \sin 2\pi t_i\ , \quad (3)$$

and the process is described by (1) provided $a > b$.

Repeated iteration of (1) generates a sequence of points $x_1 = f(x_0), x_2 = f(x_1) = f^2(x_0), \ldots$. There is a fixed point of (1) of period N if

$$x_{t+N} = x_t; \quad x_{t+j} \neq x_t\ , \quad 1 \leqslant j < N\ . \quad (4)$$

If there is a fixed point of period N, then there will be a cycle of period N, $x_0^*, x_1^*, x_2^*, \ldots x_N^* = x_0^*$. A cycle is stable if

$$\left|\left(\frac{\partial f^n}{\partial x_t}\right)_{x_t=x_0^*}\right| = \prod_{i=0}^{N-1} \left|\left(\frac{\partial f}{\partial x_t}\right)_{x_t=x_i^*}\right| < 1\ . \quad (5)$$

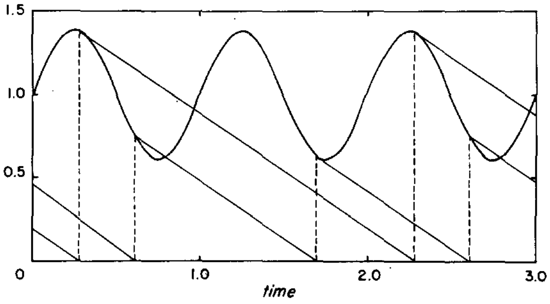

Fig. 1. A model whose phase locking properties are described by (1) and (3). There is a sinusoidally modulated threshold, and a linearly decaying activity which discontinuously jumps from a value of zero to the threshold. Phase locking is represented by the ratio of 2 integers $N:M$ where N is the frequency of the sawtooth function and M is the frequency of the sine wave. For the parameters in this figure [$\alpha = 0.69$, $\beta = 0.39$ in (3)], there is stable 1 : 2 and 2 : 2 phase locking.

Stable cycles of period N are associated with stable phase locked dynamics [6–10]. We say that there is $N:M$ phase locking if for each N cycles of the sawtooth function there are M cycles of the sinusoidal forcing function where the sawtooth resets at N distinct phases of the sinusoidal function. For the cycle $x_0^*, x_1^* \ldots x_N^* = x_0^*$, M can be readily computed

$$M = \sum_{i=1}^{N} a + b \sin 2\pi x_i^* . \qquad (6)$$

A system displays bistability if there are two different stable cycles for a single set of parameter values [8,9]. Fig. 1 illustrates bistability in which there is 1 : 2 and 2 : 2 stable phase locking for $\alpha = 0.69$, $\beta = 0.39$ in (3).

Eq. (1) displays the symmetry

$$f(0, b, 1 - x_t) = 1 - f(0, b, x_t) . \qquad (7)$$

We show elsewhere [17] that this symmetry leads to the following symmetries in the phase locking zones: (i) if there is $N:M$ phase locking for $a = 1 + \epsilon$, $0 \leqslant \epsilon \leqslant 0.5$ then there will be $N : 3N - M$ phase locking for $a = 2 - \epsilon$, and (ii) if there is $N:M$ phase locking for $a = a_0$, $1 < a_0 < 2$, then there will be $N : M + NK$ phase locking for $a = a_0 + K$ with K an integer.

The boundaries of the $1:M$ phase locking regions can be analytically computed. For $b = |a - M|$, $(\partial f/\partial x_t)_{x_t = x^*} = 1$, and thus as b is increased from zero for fixed a the period 1 solutions appear via a tangent bifurcation. As b continues to increase at fixed a, the stability of the period 1 solutions are lost via a period-doubling pitchfork bifurcation when $(\partial f/\partial x_t)_{x_t = x^*} = -1$. This occurs along the family of hyperbolae $b^2 - (M - a)^2 = \pi^{-2}$ which separate the 1 : M and 2 : 2M phase locking regions.

Additional analytic results can be obtained only for special cases. For the values $a = j$, $j = 0, 1, 2, \ldots$, there are two stable period 2 solutions of (1) for $0.5 \leqslant b \leqslant (0.25 + 1/2\pi^2)^{1/2}$ corresponding to stable $2 : 2j$ phase locking. At $b = (0.25 + 1/2\pi^2)^{1/2}$ the two stable period 2 solutions lose their stability by a period-doubling pitchfork bifurcation leading to two stable period 4 solutions corresponding to $4 : 4j$ phase locking. Analtyic results can also be obtained for $a = j + 0.5$, $j = 0, 1, 2, \ldots$. For $0 \leqslant b \leqslant \pi^{-1}2^{-1/2}$ there is a stable $2 : 2j + 1$ phase locking. At $b = \pi^{-1}2^{-1/2}$ the period 2 solution loses stability by a period doubling pitchfork bifurcation leading to a period 4 solution corresponding to $4 : 4j + 2$ phase locking.

Numerical studies have been used to determine the boundaries of some phase locking regions as a function of a and b. The analysis was aided by computing numerical estimates of the rotation number, ρ, and Lyapunov number, λ, which are defined,

$$\rho(a, b, x_0) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} a + b \sin 2\pi x_i , \qquad (8)$$

$$\lambda(a, b, x_0) = \lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N} \log \left| \left(\frac{\partial f}{\partial x_t}\right) \right|_{x_t = x_i} \qquad (9)$$

The rotation number is rational for periodic dynamics and irrational for aperiodic dynamics [12–15]. The Lyapunov number is negative for stable cycles, zero for quasiperiodic dynamics and at tangent and pitchfork bifurcations, and positive for unstable cycles and chaotic dynamics [18,19]. The numerical estimates for ρ and λ were computed over 100 iterations of (1) for several initial conditions following an initial transient of 10 iterations. We take a positive value of the Lyapunov number as an operational definition for chaotic dynamics [18,19].

Fig. 2 shows the boundaries of some of the main phase locking regions as a function of a and b. The width of some of the regions (e.g. 3 : 4 and 2 : 3) is so narrow as b increases that the width cannot be accurately represented in this figure. Thus, the cusp-like extensions of the 3 : 4 phase locking region as b increases, enclose a region whose boundaries have been collapsed into a single line in the drafting of the figure. In the labelled phase locking regions there is

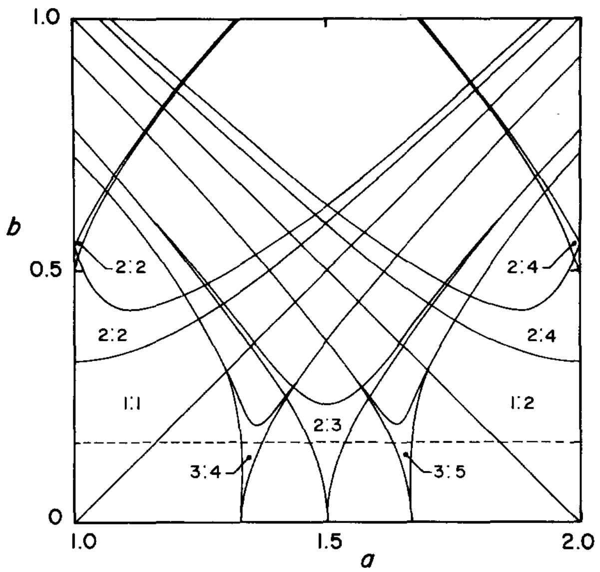

Fig. 2. Locally stable phase locking regions for (1). The dashed line at $b = 1/2\pi$ separates the regions in which (1) is a monotonic function of the unit circle ($b < 1/2\pi$) and a non-monotonic function ($b > 1/2\pi$). In the non-labelled regions are phase locked, quasiperiodic and chaotic dynamics (see text).

stable phase locking, but all initial conditions will not necessarily be attracted to the stable cycle(s).

For $0 < b < 1/2\pi$ both stable phase locking and quasiperiodic dynamics are found [12–15]. The rotation number, ρ, does not depend on x_0 and is a continuous function of a and b. As a increases at constant b, ρ increases but is piecewise constant on the set of rotational values [12–15]. The value $b = 1/2\pi$ is indicated by the dashed line in fig.2. The phase locking regions for $0 < b < 1/2\pi$ agree with the classic results from Arnold [15, fig. 78].

For $b > 1/2\pi$ the dynamics are much more complex. As b increases at fixed a, the stable phase locked regions (1 : 1, 3 : 4, 2 : 3, 3 : 5, 1 : 2) initially appear via a tangent bifurcation. As b continues to increase stability is lost via a period-doubling pitchfork bifurcation and each of the $N : M$ phase locking regions mentioned above is contiguous with a $2N : 2M$ phase locking region. Cascading period-doubling bifurcations are observed which appear similar to the period-doubling bifurcations observed in maps of the unit interval [3–5]. The boundaries of the phase locking regions intersect leading to bistability. Although numerical studies cannot guarantee that each period-doubling sequence continues to the chaotic region as b increases, we have not found situations in which this does not occur. Only the phase locking regions with the largest area in (a, b) parameter space are included in fig. 2. In a subsequent paper we will show that the other stable cycles for $0 < b < 1/2\pi$ also extend to $b > 1/2\pi$. Each one splits into two branches in a fashion similar to the splitting of the 2 : 3, 3 : 4 and 3 : 5 regions.

Since other periodically forced nonlinear oscillators can be analyzed by one dimensional Poincaré maps having the symmetry and topological properties of (1), we expect that the main features of fig. 2 will also be found in other systems. However, experimental observation of this rich structure will be difficult since many of the phase locking regions occupy extremely small regions of parameter space and are easily destroyed by noise.

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