BIOLOGY 309A: PROBLEM ASSIGNMENT #1

Due Thursday Sept. 29

Problem A

Book Problem 1.3

Problem B

Book Problem 1.22

Problem C

For each of the following, find a function $f(x)$ so that the finite-difference equation has

1. a stable fixed point
2. a stable cycle of period 3
3. a stable cycle of period 10
4. all of the above in the same function

You just have to sketch the function, not provide an algebraic formula. Since you won't be giving the algebraic formula, you will have to explain in a sentence or two how you know that the function you sketch produces a *stable* fixed point or cycle.

Problem D

You are a population ecologist studying course enrollments at universities. You notice that when classes are large, students tend not to like them and give them poor evaluations. Conversely, small classes receive good evaluations. Next year's students read these evaluations when deciding whether to enroll, and avoid enrolling in courses rated poor. Let N_t be the number of students enrolled in year t. We want to construct a finite-difference equation $N_{t+1} = f(N_t)$ that models course enrollments.

1. Draw a graph of $f(N_t)$ that models the course-avoidance behavior described above, and justify (in words) its shape.
2. Can this graph produce year-to-year alternation between high and low enrollment? Tell what conditions on $f(N_t)$ are needed so that the alternation does not decay in amplitude to zero, i.e. so that the fixed point of $f(N_t)$ is unstable.
3. A university administrator decides that fluctuating enrollments are bad, and wants to change the course evaluation process in order to reduce them. Two plans are proposed. Choose one of them, and analyze it to see whether it can be successful. In particular, assume that the function $f(N_t)$ is such that the the fixed point is unstable and tell whether the plan will change the dynamics to stablize the fixed point.

   (a) The university does not release the evaluations for more than a year, so that students make their enrollment decisions based on the evaluations from 2 years ago

$$N_{t+2} = f(N_t).$$

(b) The university averages together each year's evaluations with previous evaluations, so that the published evaluation E_t in year t is

$$E_t = \frac{G_t + E_{t-1}}{2} \tag{1}$$

where G_t is the "raw" result of the student evaluations in year t. For simplicity, let's assume that $N_{t+1} = E_t$, that is, next year's enrollment is simply the published score for the course. As before, we have that the raw evaluation is a function of the previous year's enrollment, i.e., $G_{t+1} = f(N_t)$. Substituting these into Equation 1 we get

$$N_{t+1} = \frac{f(N_t) + N_t}{2}$$

where $f(N_t)$ is the graph you sketched previously.

Problem E

Finite-difference equations can be used as computers.

As an example, let's assume that you want to calculate the square root of a number, but that your calculator does not have a square root button. We have a number x and we want to calculate $y = \sqrt{x}$. If we could write down a finite-difference equation

$$y_{t+1} = f(y_t)$$

that has a stable fixed point at $y_t = \sqrt{x}$, then if we make an initial guess of $\sqrt{x}$ that is in the basin of attraction of the fixed point, we will simply have to iterate the finite-difference equation in order to find $\sqrt{x}$ to any desired accuracy.

Consider the finite-difference equation

$$y_{t+1} = y_t - \frac{y_t^2 - x}{2y_t}. \tag{2}$$

Remember, y_t is the dynamical variable. x is just a number that stays constant.

1. Show that this has a fixed point at $y_t = \sqrt{x}$.
2. Show that this fixed point is stable.
3. Count the number of letters in your name. Use Equation 2 to find the square root of this number with an initial condition of $y_0 = 1994$.

This method for finding square roots is a specific case of a very general method, called the Newton-Raphson method, and is very widely used in scientific computing.

[Optional] A more general formula for finding the nth root of a number, $y = \sqrt[n]{x}$ is the finite-difference equation

$$y_{t+1} = y_t - \frac{y_t^n - x}{n y_t^{n-1}}. \tag{3}$$

Show that this finite-difference equation has a stable fixed point at $y = \sqrt[n]{x}$.