Biology 309a: 1995
Problem Assignment #4 Solutions

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Problem A - Book 5.4

The equations are

$$\frac{dg}{dt} = -m_1 g - m_2 h$$
$$\frac{dh}{dt} = -m_3 h + m_4 g.$$

Linearizing, to put them in the form

$$\frac{dx}{dt} = Ax + By$$
$$\frac{dy}{dt} = Cx + Dy$$

we find $A = -m_1$, $B = -m_2$, $C = m_4$, and $D = m_3$. The eigenvalues are therefore

$$-\frac{m_1 + m_3}{2} \pm \frac{\sqrt{(-m_1 + m_3)^2 - 4m_2 m_4}}{2}.$$

$\boxed{\text{a}}$ Is $m_1 + m_3$ bigger than $\sqrt{(-m_1 + m_3)^2 - 4m_2 m_4}$? If so, then both eigenvalues must have real parts less than zero. Squaring both sides, and recalling that m_1, m_2, m_3, and m_4 are all positive, we find that $m_1 + m_3$ is in fact greater than the discriminant. Since both eigenvalues have real parts less than zero, $\lim_{t \to \infty} g(t) = 0$.

$\boxed{\text{b}}$ There is an oscillatory approach to the steady state if the discriminant is negative, that is, if

$$-(m_1 + m_3)^2 - 4m_2 m_4 < 0.$$

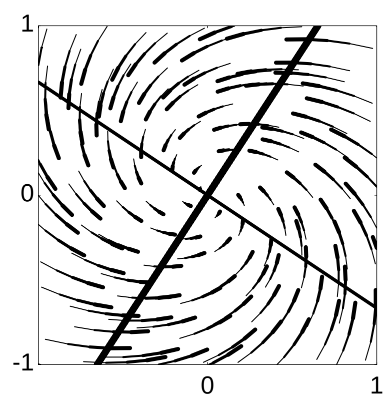
$\boxed{\text{c}}$ The g-isocline is when $dg/dt = 0$, or,

$$h = -\frac{m_1}{m_2} g.$$

The h-isocline is when $dh/dt = 0$, or,

$$h = \frac{m_4}{m_3} g.$$

Both these isoclines are lines through the origin. The g-isocline has a negative slope and the h-isocline has a positive slope.



Flow in the g, h plane. The g-isocline is the thin line, the h-isocline is the thick line.

Along the g-isocline, $dg/dt = 0$, so only h can change. Therefore, the flow is vertical. We need only decide if it is upwards or downwards. The direction of the vertical flow depends on which side of the h-isocline we are on: to one side, the flow is up, and to the other it is down. We have to decide which side is up. When $g > 0$ and $h = 0$, we have $dh/dt = m_4 g > 0$, so the flow is up on that side of the h-isocline. When $g < 0$ and $h = 0$, we have $dh/dt = m_4 g < 0$ so the flow is down. A similar argument shows that the horizontal flow on the h-isocline is leftward when $g > 0$ and rightward when $g < 0$.

Problem B — Book 5.16

The limpet and seaweed equations are:

$$\begin{aligned} \frac{ds}{dt} &= s - s^2 - sl = f(s,l) \\ \frac{dl}{dt} &= sl - l/2 - l^2 = g(s,l). \end{aligned}$$

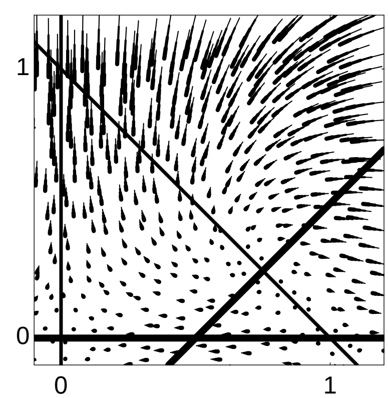
$\boxed{\text{a}}$ The s-isocline is where $0 = s - s^2 - sl$, implying the two lines

$$s = 0 \text{ or } l = 1 - s.$$

The l-isocline is where $0 = sl - l/2 - l^2$, implying

$$l = 0 \text{ or } l = s - 1/2.$$

Steady states occur where the s- and l-isoclines intersect each other. The steady states occur at three places, $(s = 0, l = 0)$, $(s = 1, l = 0)$, $(s = 3/4, l = 1/4)$.



Flow in the s,l plane. The s-isocline is the thin line, the l-isocline is the thick line.

$\boxed{\text{b}}$ Linearizing, we get

$$\begin{array}{ll} A = 1 - 2s - l & B = -s \\ C = l & D = s - 1/2 - 2l \end{array}.$$

At $(s = 1, l = 0)$ this gives $A = -1$, $B = -1$, $C = 0$, and $D = 1/2$. The eigenvalues are therefore $-\frac{1}{4} \pm \frac{3}{4}$. One eigenvalue is positive and one is negative, so the steady state is a SADDLE.

At $(s = 3/4, l = 1/4)$, we have $A = -3/4$, $B = -3/4$, $C = 1/4$, and $D = -1/4$, giving eigenvalues of $-\frac{1}{2} \pm i\frac{\sqrt{1/2}}{2}$. This is a FOCUS.

$\boxed{\text{c}}$ The flow can be easily sketched along the isoclines. Along the diagonal l-isocline, we have $l = s - 1/2$ and we know the flow is horizontal. Along that isocline, we find

$$\frac{ds}{dt} = s - s^2 - s(s - 1/2) = \frac{3}{2}s - 2s^2.$$

This tells us that for $s > 3/4$, $\frac{ds}{dt} < 0$ and so the flow is leftward. For $s < 3/4$ the flow is rightward along the isocline.

Arguing similarly, along the s-isocline given by $l = 1 - s$, the flow is vertical and $\frac{dl}{dt} = \frac{l}{2} - 2l^2$. This means that the flow is downward when $l > 1/4$ and upward when $l < 1/4$.

$\boxed{\text{d}}$

i) $s(0) = 0$, $l(0) = 0$ is a fixed point. Although it is unstable, if we start there, we stay there.

ii) $s(0) = 0$, $l(0) = 15$ results in our moving to the origin. Note that when $s = 0$ then $\frac{ds}{dt} = 0$, so s can never change from zero. Since $\frac{dl}{dt} < 0$ for $s = 0$, l will eventually decay to zero.

iii) $s(0) = 2$, $l(0) = 0$. From here, we move to the saddle. Note that when $l = 0$, $\frac{dl}{dt} = 0$, so l will stay zero.

iv) $s(0) = 2$, $l(0) = 15$ results in moving to the stable fixed point at $(s = 3/4, l = 1/4)$.

Problem C — Book 5.9

$\boxed{\text{a}}$ $\frac{d(N_1+N_2)}{dt} = \alpha\left(\frac{N_2}{V_2} - \frac{N_1}{V_1}\right) + \alpha\left(\frac{N_1}{V_1} - \frac{N_2}{V_2}\right) = 0.$

$\boxed{\text{b}}$ There is a steady state when $\frac{N_2}{V_2} = \frac{N_1}{V_1}$. Since $N_1 + N_2 = M$, we have $M - N_1 = \frac{V_2}{V_1}N_1$ implying

$$\boxed{N_1 = \frac{MV_1}{V_2+V_1}} \text{ and } \boxed{N_2 = M - N_1 = \frac{V_2 M}{V_2+V_1}}.$$

$\boxed{\text{c}}$

$$\ddot{N}_2 = \alpha\left(\frac{\dot{N}_1}{V_1} - \frac{\dot{N}_2}{V_2}\right) = -\frac{\alpha\dot{N}_2}{V_2} + \frac{\alpha^2}{V_1}\left(\frac{N_2}{V_2} - \frac{N_1}{V_1}\right)$$

but

$$\frac{N_1}{V_1} = \frac{1}{\alpha}\dot{N}_2 + \frac{N_2}{V_2}.$$

Substituting this in for $\frac{N_1}{V_1}$, we arrive at

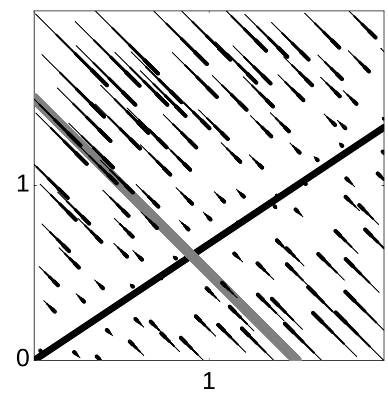
$$\ddot{N}_2 = -\alpha\dot{N}_2\frac{V_1+V_2}{V_1V_2}.$$

$\boxed{\text{d}}$ The characteristic equation of the result found in (c) is

$$\lambda^2 - \alpha\frac{V_1+V_2}{V_1V_2}\lambda = 0,$$

giving the two eigenvalues

$$\lambda_1 = 0 \text{ and } \lambda_2 = \alpha\frac{V_1+V_2}{V_1V_2}.$$



Flow in the N_1, N_2 plane. Both isoclines are the same, shown by the thick, black line. The flow is zero at the isoclines, because anywhere the isoclines intersect, there is a fixed point. The thick, gray line shows $M = N_1 + N_2 = 1.5$. Note that the flow is such that if the initial condition is on this line, the state will stay on this line. This is how the system conserves mass, as shown in part (a).

The solution is therefore $N_2(t) = Ke^{\lambda_1 t} + Be^{\lambda_2 t} = K + Be^{\lambda_2 t}$. The initial condition is $N_(0) = 0$ so $K + B = 0$ and we have

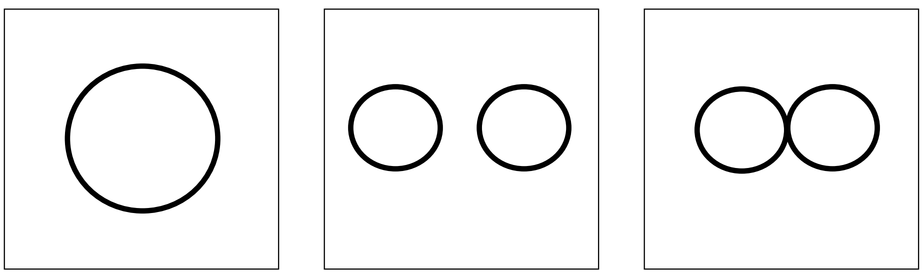
$$N_2(t) = K(1 - e^{\lambda_2 t}).$$

$\boxed{\text{e}}$ As $t \to \infty$, $N_2 = \frac{V_2 M}{V_2 + V_1}$, implying

$$\boxed{K = \frac{V_2 M}{V_2 + V_1}} \text{ and } \boxed{\gamma = -\alpha \frac{V_1 + V_2}{V_1 V_2}}.$$

$\boxed{\text{f}}$ It takes approximately 2 minutes to reach half of the way to the final value, so $e^{-2\gamma} = 1/2$, giving $\gamma = -\frac{\ln 2}{2}\text{min}^{-1}$.

Problem D



(a) (b) (c)

At the intersection point in (c), the flow would have to be in 2 directions. This is only possible if the flow is zero, in which case the intersection point is a fixed point.

Problem E

Answering the personal questions, I find $A = 5$, $B = 6$, $C = 6$, and $D = 35$, giving eigenvalues

$$\boxed{\lambda = 20 \pm 13.75}.$$

(Keep in mind that the B in the eigenvalue formula Eq. 5.13, is the negative of the B in this problem, since the first equation is $\dot{x} = Ax - By$.)

The linear equation given in the problem is a general equation near a steady state, where the linear approximation is valid. (If we assume that the parameters are all positive, then the equation is no longer general.) As always, exponential growth is valid only very close to the steady state. So, we can tell the *Times* that our theory has little to say about the fate of the Earth, since we are far away from the black hole.

Bonus

We will modify the Lotka-Volterra equations to include Verhulst growth for the predator, y. This gives

$$\begin{aligned} \frac{dx}{dt} &= \alpha x - \beta x y = f(x,y) \\ \frac{dy}{dt} &= \gamma x y + k y - b y^2 = g(x,y) \end{aligned}$$

The x-isocline occurs when $\alpha x - \beta x y = 0$, implying

$$\boxed{x = 0} \text{ or } \boxed{y = \alpha/\beta}.$$

The y-isocline occurs when $\gamma x y + k y - b y^2 = 0$, giving

$$\boxed{y = 0} \text{ or } \boxed{y = \tfrac{\gamma x + k}{b}}.$$

There are two fixed points, at $(x = 0, y = 0)$ and $(x = 0, y = k/b)$. When $\frac{k}{b} < \frac{\alpha}{\beta}$, there is a third fixed point at $(x = \frac{\alpha - k\beta}{\gamma\beta},\ y = \frac{\alpha}{b})$.

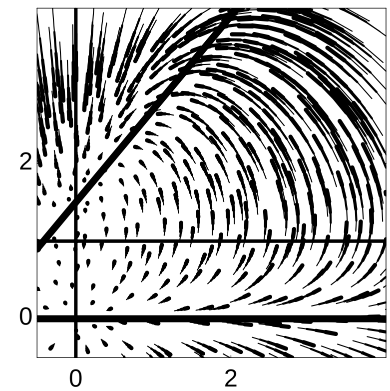
In order to look at the stability of these fixed points, we linearize, as always, getting

$$\begin{array}{ll} A = \frac{\partial f}{\partial x} = \alpha - \beta y & B = \frac{\partial f}{\partial y} = -\beta x \\ C = \frac{\partial g}{\partial x} = \gamma y & D = \frac{\partial g}{\partial y} = \gamma x + k - 2by \end{array}.$$

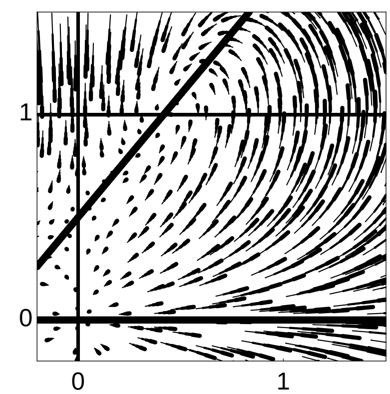
The eigenvalues are

$$\boxed{\lambda_1 = \alpha - \tfrac{k\beta}{b}} \text{ and } \boxed{\lambda_2 = -k}.$$

If $\alpha/\beta < k/b$, then the fixed point at $(x = 0, y = k/b)$ is stable, and extinction of the prey is the outcome for any initial condition where $y(0) > 0$.



Flow in the x, y plane for the modified Lotka-Volterra equations. $\alpha = 1$, $\gamma = 1.2$, $\beta = 1$, $k = 1.5$, and $b = 1$. The thin lines are the x-isoclines, and the thick lines are the y-isoclines. The flow is towards a steady state where the prey are extinct.



$\alpha = 1$, $\gamma = 1.2$, $\beta = 1$, $k = 0.5$, and $b = 1$. The flow is towards a steady state where both predator and prey exist.