

Orbital Mechanics: The Dynamics of Planetary and Spacecraft Motion

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1. Circular Orbits

Circular orbits apply to spacecraft in orbits with constant altitude above planets, and are also good approximations to planets orbiting the Sun. Since the spacecraft or planet is traveling in a circle at constant speed, they “feel” a centrifugal force outward of

$F_{centrifugal} = \frac{mv^2}{R}$. This centrifugal force is balanced by

gravity pulling the body inward: $F_{gravity} = \frac{GMm}{R^2}$, where M

is the mass of the primary (the body being orbited), m is the mass of the satellite or spacecraft, and G is the universal constant of gravitation ($G = 6.67 \times 10^{-11} \text{ m}^3/\text{kg s}^2$). Equating these two forces, which must balance if the satellite is to continue in a circular

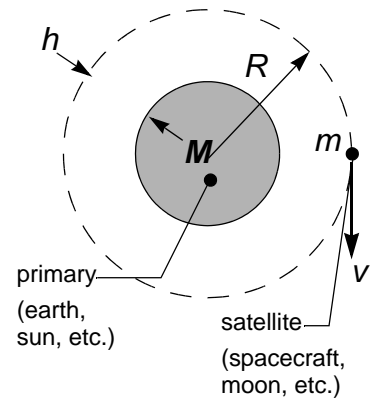
path, we obtain the velocity for a circular orbit: $V = \sqrt{\frac{MG}{R}}$ (Eq. 1). Two examples are

low Earth orbit (orbit of Space Shuttle or Space Station) with an altitude of 200 km:

$$V_{LEO} = \sqrt{\frac{M_{earth}G}{R_{earth} + h_{satellite}}} = \sqrt{\frac{(5.974 \times 10^{24} \text{ kg}) \left(6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg s}^2} \right)}{6378 \times 10^3 + 200 \times 10^3 \text{ m}}} = 7780 \text{ m/s}$$

and the orbit of the Earth about the Sun:

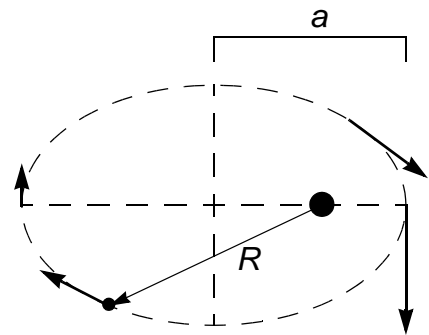
$$V_{earth} = \sqrt{\frac{M_{sun}G}{R_{earth-sun}}} = \sqrt{\frac{(1.989 \times 10^{30} \text{ kg}) \left(6.67 \times 10^{-11} \frac{\text{m}^3}{\text{kg s}^2} \right)}{1.496 \times 10^{11} \text{ m}}} = 29,800 \text{ m/s}$$



Likewise, Mars has an orbital velocity of $V_{mars} = 24,100$ m/s. Since it is further out than the Earth, the Sun's gravity does not pull as strong and so Mars does not need to go as fast.

2. Elliptical Orbits

Johannes Kepler showed in 1609 that planets actually orbit in ellipses, not circles, with the primary body located at one focus of the ellipse. We can approximate the orbits of planets as circles, as we did in Section 1. But very elliptic orbits are encountered with comets and spacecraft transferring from one circular orbit to another, and must be treated separately from circular orbits. The analogous equation to Eq. 1 above is the *vis-viva* equation:



$$v = \sqrt{GM\left(\frac{2}{R} - \frac{1}{a}\right)} \quad (\text{Eq. 2})$$

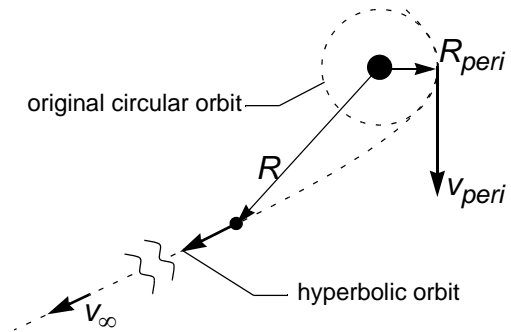
where R is the distance from the satellite to the primary and a is the semi-major axis of the ellipse. At the maximum distance from the primary (the “high point” of the orbit, or “apogee/apoapse”), $r_{max} = a(1 + e)$, where e is eccentricity. At this point, velocity will be minimum. Likewise, at the point of closest approach (the “low point” or “perigee/periapse”), $r_{min} = a(1 - e)$, and velocity is a maximum. Note that for circular orbit, $e = 0$ and we just get the equation for a circular orbit back. The time it takes to go around an elliptical orbit is:

$$T = 2\pi \sqrt{\frac{a^3}{GM}}$$

3. Hyperbolic Orbits

If we want to send a spacecraft to another planet, we need to first escape Earth's gravity entirely. We do this by launching the spacecraft at a velocity greater than the escape velocity:

$$V_{escape} = \sqrt{\frac{2MG}{R_{peri}}}. \text{ Note that the escape velocity is}$$



just $\sqrt{2}$ times the circular velocity. So, the velocity necessary to escape Earth's gravity entirely is: $\sqrt{2} V_{LEO} = \sqrt{2} \cdot 7780 \text{ m/s} = 11,000 \text{ m/s}$. Usually, spacecraft on interplanetary missions are launched with velocities greater than this so that when they escape Earth's gravity, they still have some velocity left over. The path such an escaping satellite travels on is a hyperbola, again with primary body (planet) at the focus of the hyperbola. When the spacecraft has escaped Earth's gravity by traveling very far away, the velocity left over is called the hyperbolic excess velocity v_{∞} :

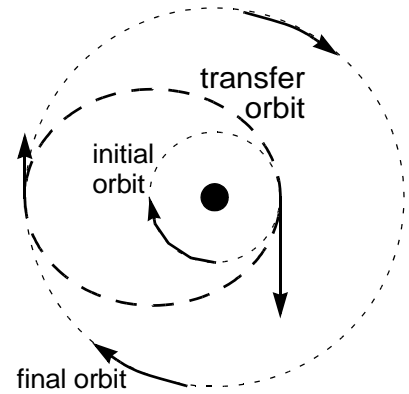
$$v_{\infty}^2 = v_{peri}^2 - \frac{2MG}{R_{peri}} = v_{peri}^2 - v_{escape}^2$$

where v_{peri} was the velocity at the periapsis (point nearest to Earth) and v_{∞} denotes the hyperbolic excess velocity left over when the spacecraft reaches infinite distance from earth. Of course, no spacecraft ever goes that far, but after traveling several hundred times the Earth's radius (far beyond the Moon's orbit), the spacecraft has *effectively* escaped and retains a velocity v_{∞} relative to Earth.

4. Transfer Orbits

Spacecraft do not go in a straight line from one planet to another; we do not have rocket engines powerful enough to do that. Instead, a rocket fires for just a few minutes, then the vehicle coasts for days (or months or even years) until it reaches its destination. Since there is no other force than gravity acting on the spacecraft, it is in an orbit just like a natural body, and all the same equations apply.

In 1925, the German Walter Hohmann demonstrated that the transfer orbit that requires the smallest change in velocity (and hence the least rocket fuel) is an elliptical orbit that just touches the original and final desired orbits. Note that this means we launch our spacecraft to Mars when Mars is near the opposite side of the Sun, at its furthest distance from Earth!



For example, in going from Earth to Mars, the spacecraft will travel on an elliptical orbit with a periaipse (distance closest to the Sun) equal to the Earth's orbital radius and an apopase (distance farthest from the Sun) equal to Mars's radius. From the vis-viva equation (Eq. 2), the velocity required at Earth to be on this transfer orbit is

$$v_{earth\ depart} = \sqrt{GM_{sun} \left(\frac{2}{R_{earth\ orbit}} - \frac{2}{R_{earth\ orbit} + R_{mars\ orbit}} \right)}$$

$$v_{earth\ depart} = \sqrt{GM_{sun} \left(\frac{2}{1.496 \times 10^{11} m} - \frac{2}{1.496 \times 10^{11} m + 1.524 \times 1.496 \times 10^{11} m} \right)}$$

which gives $v_{earth\ depart} = 32,700$ m/s and the velocity it will have when it gets to Mars is $v_{mars\ arrival} = 21,500$ m/s. In Section 1 we found the orbital velocity of Earth to be $v_{earth} = 29,800$ m/s about the Sun. So, a spacecraft leaving Earth must increase its velocity by an amount $v_{earth\ depart} - v_{earth} = 32,700 - 29,800 = 2,900$ m/s.

This change in velocity is the difference in the spacecraft velocity and the Earth's orbital velocity. But before that, the spacecraft has to escape Earth along a hyperbolic transfer. In other words, viewed relative to Earth, the spacecraft is escaping to infinity with a final velocity of 2,900 m/s, but relative to the sun it has an increase in velocity to $29,800 + 2,900$ m/s = 32,700 m/s, which puts it on the transfer orbit to Mars. The increase in velocity that it must have in low earth orbit in order to have the desired

hyperbolic excess velocity of $v_\infty = 2,900$ m/s is given by Eq. 3, re-written in a slightly different form here:

$$\Delta V_{LEO} = V_{LEO} \left\{ \left[2 + \left(\frac{v_\infty}{V_{LEO}} \right)^2 \right]^{\frac{1}{2}} - 1 \right\}$$

Here “ ΔV ” refers to the *change* in velocity, pronounced “*Delta Vee*” by aerospace engineers. So, to leave Earth with the required $v_\infty = 2,900$ m/s, we need a ΔV_{LEO} of 3,600 m/s if we start from low Earth orbit with orbital velocity 7,780 m/s.

Once we get to Mars, we require another change in velocity to match Mars’ orbital velocity about the sun: $\Delta V_{arrival} = V_{mars\ arrival} - V_{mars} = 21,500$ m/s – 24,100 m/s = – 2600 m/s. But now spacecraft can get this change in velocity for “free” by using the Martian atmosphere to “break” the spacecraft. This maneuver requires very little rocket fuel, but does require that we bring along a heat shield so our spacecraft does not burn up. This technique is called “aerobreaking” or “aerocapture.” It is now being used on Mars missions (not without some incidents, however), and we can count on using this technique to save on fuel for future missions.

Note that the spacecraft has to speed up to get away from the Earth and move outward in the solar system. If you want to send a spacecraft inward (to Venus or Mercury), there is also a ΔV required to slow the spacecraft down so it will “fall” inward. You don’t “fall” down for free, however, and the ΔV required for a Hohmann transfer to Mercury is more than the one required for Mars! ($\Delta V_{LEO\ to\ mercury} = 5,600$ m/s.) While this seems like a lot, note that it is not nearly as great a velocity as needed to get into low Earth orbit in the first place (ΔV to reach low Earth orbit = 7800 m/s). “*Reach low orbit and you’re halfway to anywhere in the Solar System.*” — Robert A. Heinlein.

We can now have the necessary tools to determine the total ΔV required to get around the solar system. This ΔV then determines how much fuel we must bring to run our rockets, and in turn the cost of the mission.

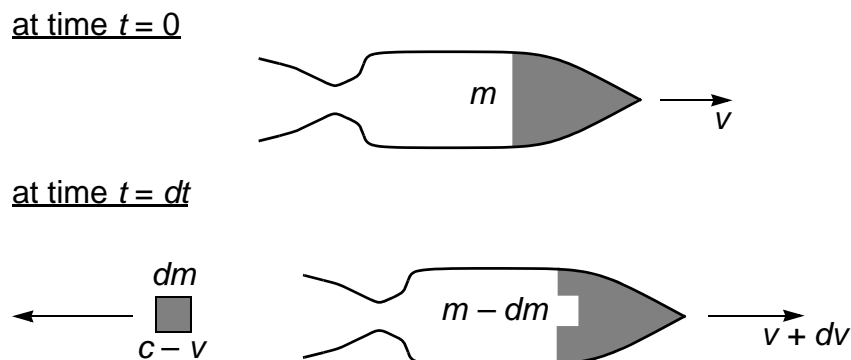
Reference: Prussing, J.E., and Conway, B.A., *Orbital Mechanics*, Oxford University Press, New York, 1993. ISBN: 0-19-507834-9

Rocket Propulsion: The Engines of Spaceflight

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The ΔV necessary to change the trajectory of a spacecraft comes from rockets. Rockets for spacecraft are very compact and very powerful. Typical rockets can generate 100 times or more of their own weight in thrust and typically occupy a small volume of the total vehicle. Unfortunately, they are very thirsty as we need to supply them with both fuel (hydrogen, kerosene, etc.) and oxidizer (oxygen), since there is no air in space to burn with the fuel. This means that the majority of the mass of most space vehicles at launch is the fuel and oxidizer (collectively called the “propellant”). This handout discusses how we estimate how much fuel we need for a given type of rocket engine.

Rockets work by throwing mass out the back. While a firing rocket engine looks like an explosive event with hot, high pressure gas spewing out the nozzle, the hot, high pressure combustion products are actually only found inside the engine. As the gas expands out the rocket nozzle, it cools and expands to low pressure (approximately equal to ambient) and low temperature. The thermal energy in the gas gets converted into kinetic energy, so the exhaust has very high velocity (supersonic, typically several thousands of kilometers per hour). The operation of a rocket is shown here:.



Here, the rocket of mass m is originally moving at velocity v . After throwing a small mass dm out the back at velocity c relative to the rocket, its velocity had increased by dv and

mass has decreased by dm . Writing down the conservation of linear momentum (the total of mv is a constant):

$$mv = (m - dm)(v + dv) - dm(c - v)$$

Expanding this out and tossing out higher order terms, we are left with:

$$dv = c \frac{dm}{m}$$

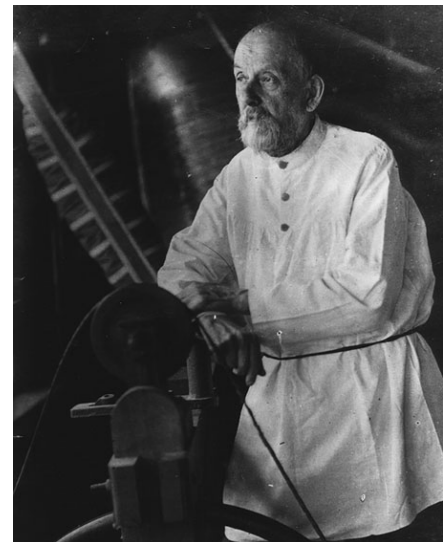
We can integrate this differential equation over time for a large number of chunks thrown out the back and find the total change in velocity:

$$\Delta V = c \ln\left(\frac{m_{initial}}{m_{final}}\right) \quad (\text{Eq. 1})$$

Thus, the total change in the rocket's velocity is the logarithm of the initial mass (fully fueled) rocket divided by the final mass of the rocket (empty of fuel) times the velocity at which mass is thrown out the back. Thus, we want to throw mass out the back as fast as we can, maximizing c . The difference between $m_{initial}$ and m_{final} is just the mass of the propellant that was consumed: $m_{propellant} = m_{initial} - m_{final}$.

Equation 1 is the Rocket Equation, sometimes known as the Tsiolkovski Equation, after the Russian school teacher who first formulated it circa 1900. Rocket engineers usually divide the exhaust velocity c by $g_0 = 9.81 \text{ m/s}^2$. This gives a strange parameter called specific impulse (abbreviated $I_{sp} = c/g_0$) which has units of seconds. The rocket equation expressed using I_{sp} looks like:

$$\Delta V = I_{sp} g_0 \ln\left(\frac{m_{initial}}{m_{final}}\right)$$



*Earth is the cradle of Humanity,
but one cannot remain in the cradle forever...*
—K. Tsiolkovski

I_{sp} is the fuel efficiency of a rocket engine burning a particular fuel. The H_2/O_2 engine of the Space Shuttle has a very high I_{sp} of 455 sec. That means that 1 kg of propellant ($H_2 + O_2$) can provide 1 Newton of thrust for 455 seconds. Below are some typical fuels and engines and their I_{sp} .

Table 1: Rocket Engines

Rocket	Fuel	I_{sp}
Space Shuttle Main Engine	H ₂ /O ₂	455 sec (vacuum) 420 sec (sea level)
Solid Rocket	nitrocellulose/ammonium perchlorate/aluminum/etc.	220 – 300 sec
Liquid Storable	N ₂ H ₄ /N ₂ O ₄	300 – 320 sec
Mars Direct Earth Return Engine	CH ₄ /O ₂	380 sec
Nuclear Thermal Rocket	H ₂ heated by U/Pu reactor	800 – 1,000 sec
Ion Engine	ionized cesium, mercury, xenon	3,000 – 10,000 sec

I_{sp} is not the whole story, however, because it is only the fuel efficiency and not the thrust. Some engines, such as ion engines, have fantastic I_{sp} 's but have such small thrust ($\ll 1$ N), they could not lift themselves off the ground. But ion engines can run for a long time, and the thrust steadily adds up. The recent NASA mission Deep Space 1 demonstrated that ion engines work in space (they have been working in the lab for more than 40 years, but no one had actually tried one in space until now), so they may see use in future missions.

Note that we can solve the rocket equation for the mass of propellant required to accomplish a ΔV

$$m_{propellant} = m_{empty} \left(\text{Exp} \left[\frac{\Delta V}{g_0 I_{sp}} \right] - 1 \right)$$

Note also that the propellant required depends *exponentially* on the ratio of the velocity change ΔV to specific impulse I_{sp} . This is why aerospace engineers expend so much effort on minimizing ΔV and maximizing I_{sp} .

As an example of rocket propulsion, if we want to launch something into low Earth orbit, we have already seen that we need 7,900 m/s of velocity. Using the best chemical fuel available (H₂/O₂), we see that the mass ratio:

$$\Delta V = I_{sp} g_0 \ln\left(\frac{m_{initial}}{m_{final}}\right)$$

$$7900 \text{ m/s} = 420 \text{ s} \times 9.81 \frac{\text{m}}{\text{s}^2} \ln\left(\frac{m_{initial}}{m_{final}}\right)$$

$$\frac{m_{final}}{m_{initial}} = 0.15$$

If the final mass is only 15%, then the initial mass of the rocket must be 85% fuel. The other 15% is the rocket engine, the payload (astronauts, satellites, scientific instruments) and huge fuel tank required to hold the other 85% fuel mass. Building a flight-worthy structure like this is so difficult that no one has ever done it (but a number of people have attempted it, such as the recently cancelled NASA X-33 program). The usual way to launch into orbit is by stages: one rocket launches another rocket which launches a third. This is not only very expensive (typical launch vehicles are \$100,000,000) but also inefficient (the useful payload ends up being only 1% of the lift-off mass).

We can use the basic rocket equation to assess the feasibility of any proposed space mission. The total ΔV require by orbital dynamics determines how much fuel the spacecraft must bring along. The total mass of fuel and spacecraft that must be launched into low Earth orbit determines the cost of the mission (\$10,000/kg is the current market rate for putting mass into low Earth orbit).

We can also appraise the likelihood of more grandiose missions, such as an interstellar mission to nearby stars. Note that the best ion engine available today ($I_{sp} = 10,000 \text{ sec}$) with a very elegantly structured spacecraft (99% propellant, 1% everything else), would be capable of a velocity of 460 km/s. That would be the fastest possible man-made object. But at this speed, it would *still* take about 3000 years to reach the nearest star (Alpha Centauri, at 4.3 light years away). So, we are going to have to limit our hands-on explorations to the solar system for the time being.

Reference: Sutton, G.P., *Rocket Propulsion Elements: An Introduction to the Engineering of Rockets*, 6th ed., John Wiley & Sons, 1992.
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