3. TELEPHONE WAITING TIMES

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1. Formulating the problem; how to reach the solution.

For some years, all the experts — particularly, perhaps, in Denmark — have been aware that the application of the theory of probabilities constitutes the only possible way of attaining fully rational methods in telephone administration. This holds good with respect to the exploitation of lines and the utilization of the work of operators, and it is especially valid for the newest, more or less automatic telephone systems. I have treated some of the problems of primary importance in this connexion in an article in "Elektroteknikeren", 1917 (and later in "Elektrotechnische Zeitschrift", 1918, and "The Post Office Electrical Engineers' Journal", 1918), in which, however, I have omitted — for the sake of brevity — some of the proofs, and stated only the resulting formulae and numerical expressions. I shall mention only one important problem here, *viz.* that of finding the probability that the delay in answering, or waiting time, shall not exceed a given quantity z, expressed as a function of z. The given quantities, then, are: — The number of available lines, x; the duration of the call, t; and the intensity of traffic, y (*i. e.* the average number of conversations proceeding simultaneously, or, in a different wording, the average number of calls during the time t. It is presupposed that $y < x$; also, that a calling subscriber who cannot be connected at once because all x lines are occupied, will always wait — possibly in a "queue" with other waiting subscribers — until he gets through. The duration of the calls t is here assumed to be constant; this assumption holds good with respect to trunk calls, but is less accurate in the case of local calls, the latter generally being of rather variable duration which gives rise to a problem of a kind somewhat different from the one we propose to deal with here. For convenience, the unit of time should either be considered equal to the duration of calls, or it should be chosen in such a way that there will be an average of 1 call per unit of time; the latter method is the one preferred here, and thus $t = y$.

The solution of our problem can be reached, or, at least, approached in rather different manners; as a rule, the special case of $x = 1$ (one line) will be found easier to handle than the general case. For instance, a differential equation can be derived,

$$\binom{x}{0} f(z) - \binom{x}{1} f'(z) + \binom{x}{2} f''(z) - \ldots . \binom{x}{x} f^{(x)}(z) = f(z - t);$$

$f(z) = 0$ for all negative values of z being known in advance, it is possible, by integration of the above, to determine the variations of the function, first from $z = 0$ to $z = t$, then from $z = t$ to $z = 2\,t$, &c. The determination of the integration constants, however, will cause difficulties; everything works out smoothly only in the special case of $x = 1$, as further described in my article in "Nyt Tidsskrift for Matematik", 1909, where this case is treated in the indicated manner.

Instead, an integral equation may be employed, *viz.*

$$f(z) = \int_0^\infty f(z + u - t) \frac{u^{x-1}}{(x-1)!} e^{-u} du$$

which immediately leads to a (sometimes) rather convenient numerical solution, but hardly to an explicit mathematical solution.

In the following we shall move along a quite different path, beginning with the introduction of a set of constants: $a_0, a_1, a_2, \ldots . a_{x-1}$; these are functions of y, or, if you like, of α, α denoting the ratio of y to x. These constants are determined, as we shall see, by inference from some elementary considerations leading to the employment of certain infinite series, all the terms of which are values of the function $e^{-y} \cdot \frac{y^x}{x!}$, and in a tabular representation of the function, the terms of each series will be placed along one or another oblique line, and distributed at equal intervals. *K. Pearson's* collection of tables contains such a table, although for positive values of y only; a similar table comprising negative values of y is given below in the appendix[1]). It should further be noted, with respect to the function $e^{-y} \cdot \frac{y^x}{x!}$, that its significance for the present problem, and for several other ones as well, depends on the following important theorem, the mathematical contents of which was found by *Poisson:* The probability of an arbitrary number of calls (x) being originated during an interval of time with an average number of calls y,

[1]) This table is omitted in the present reprint, as it is identical with Table 2, p. 137, to which the reader is referred.

is equal to $e^{-y}\frac{y^x}{x!}$. I give a simple proof of this theorem in the appendix below.

In some cases, the determination of the constants $a_0, a_1, a_2 \ldots . a_{x-1}$ in the manner indicated is very useful; in other cases, such as when α is great (nearly 1), it is very unpractical, however, as the series then are slowly convergent. We shall therefore also give the determination of the constants in a different and more aesthetic form. We introduce a set of auxiliary terms, most often imaginary, $\beta, \gamma, \ldots .$, their total number being x when α is included; they are determined by means of a certain transcendental equation in which they are roots. By using a theorem set forth by Mr. *J. L. W. V. Jensen*, Telephone Engineer-in-Chief, Ph. D., the infinite series mentioned can then be summed. The solution of our problem will then appear in a simple and convenient form.

In the following I shall pass in view the two special cases explicitly and uniformly, first $x = 1$, *i. e.* 1 line (in sections 2—6), then $x = 2$, *i. e.* 2 lines (in sections 7—11); consequently, I have considered it unnecessary to account for the proof of the general case expressly.

2. The simpler case of $x = 1$; definition of a_0.

We understand by a_0 the probability that there will be no waiting time after an arbitrary call. Here we have immediately $a_0 = 1 - \alpha$, a_0 being the probability that the line is unoccupied, and α being the probability that it is occupied.

3. The table and the oblique lines.

When x and y both are variable, the table of the Poisson function $e^{-y}\frac{y^x}{x!}$ will fill a plane; we may begin with placing an x-axis and a y-axis in the plane (*e. g.* the x-axis downwards, and the y-axis pointing to the right), and then inscribe each separate value of the function as near as possible to the point determined by the coordinates x and y. Incidentally, we shall have to deal with integral values of x only; and if desired, the negative values of x can be omitted, the function here being 0. Now, we imagine a certain set of oblique lines being laid in the plane, all having the directional coefficient α. On each line we select a number of equally spaced points, each interval corresponding to an increase of 1 in the abscissa, and of α in the ordinate. The sum of the functional values under consideration is denoted by the letter σ, to which is added as indices the coordinates of one of the points, the situation of all the other points being also given hereby. If this point is situated on the x axis, however, the second index — which is 0 — may be omitted, for the sake of brevity. We permit these series to go on infinitely in both directions or, if you choose, in the one direction, and in the other direction until the terms automatically become equal to 0. If only those of the terms corresponding to points with positive ordinates be included in the series, the sum is denoted by s; if only the other terms, *viz.* those corresponding to points with negative ordinates (and 0), be included, the sum is denoted by r. In both cases are added indices, as previously mentioned. Thus, we have always

$$r_{x,y} + s_{x,y} = \sigma_{x,y} .$$

In many cases σ and s are identical and $r = 0$, *viz.* when the oblique line intersects the negative part of the x-axis. The convergence of the series is easily realized.

4. Relations concerning a_0.

Regardless of the fact that we have already found the value of a_0 it will now be useful to prove the following relations:—

$$\left.\begin{aligned} a_0 &= 1 - a_0 s_0 \\ 0 &= 1 - a_0 s_{-1}, \end{aligned}\right\} \qquad (1\text{—}2)$$

where, in accordance with the foregoing,

$$\left.\begin{aligned} s_0 &= e^{-a}\frac{a^1}{1!} + e^{-2a}\frac{(2a)^2}{2!} + \ldots \\ s_{-1} &= e^{-a}\frac{a^0}{0!} + e^{-2a}\frac{(2a)^1}{1!} + \ldots \end{aligned}\right\} \qquad (3\text{—}4)$$

As we shall see later, the two equations (1—2) can be given a different form by introducing the sums σ instead of the sums s; but we will prove them first in the above form.

The equation (1) can be proved as follows: By considering in detail all the cases where an arbitrary call suffers a waiting time, it will be seen that the cases can be distributed, or arranged in groups, thus:—

1) During the preceding time interval of duration t (or a) there was 1 call

2) - - - - - - - $2t$ - were 2 calls

3) - - - - - - - $3t$ - - 3 calls,

etc.

An infinite number of groups is obtained; considering, however, that the probabilities in question form a convergent series, there can be no doubt that the aggregate probability $1 - a_0$ sought-after really exists in the form of a certain limit value; a similar remark could be made at several points in the following. Care should be taken, in the arranging in groups mentioned, that no case be placed under two different groups; to avoid uncertainty in this respect we will decide upon always preferring the group with the higher number to that with the lower number. Agreement with this is found in that, in group no. 1 above, 1 call is stated (*i. e.*, just one call, and no more), and the following groups are in analogy with this; but the cases which, accordingly, should be included must now be sifted further. It is easily seen that the probability that a case really belongs under the group where it has been placed temporarily, is identical with the probability that an arbitrarily chosen call will not have to suffer a waiting time; in other words, it is equal to a_0. For, if we suppose that a case has been put, temporarily, under (*e. g.*) group 3, then we know that there were 3 calls during the preceding time interval of $3\,t$; but that is all we know. We must then take the point of time that is $3\,t$ previous to the call and, from there, seek further back in time; first an interval t, to see whether 1 call can be found here; then an interval $2\,t$, to see whether 2 calls can be found here; *&c.* We must, thus, undertake the same investigation — although starting from a different point of time — as when we recently began enumerating the cases leading to a waiting time. — Accordingly, we get

$$a_0 = 1 - a_0\left(e^{-a}\,\frac{a^1}{1!} + e^{-2a}\,\frac{(2\,a)^2}{2!} + \ldots\right)$$

or, shorter, $$a_0 = 1 - a_0 s_0, \qquad q.\ e.\ d.$$

The equation (2) can be proved in a quite similar manner; we shall not dwell on that, however, as equation (1) strictly speaking will suffice. By inserting σ instead of s, the appearance of equations (1) and (2) becomes simpler and more uniform, *viz.*

$$\left.\begin{aligned} 1 &= a_0\sigma_0 \\ 1 &= a_0\sigma_{-1}. \end{aligned}\right\} \qquad (5\text{—}6)$$

The significance of these two equations (their number could easily be increased) is, for the present, that a_0 can be found by means of either of them (we leave out of account that we have already found a_0 in a simpler way here where $x = 1$). But they are, as a matter of fact, significant in another respect also, which will be dealt with later.

5. The summation of the infinite series.

The infinite series σ, as employed in the above, can be summed by means of a theorem by *Jensen* (Acta mathematica XXVI, 1902, p. 309, formula 7). With slightly altered denotations, the theorem reads:

$$\frac{1}{1-\alpha} = e^{-a} \cdot \frac{a^0}{0!} + e^{-(a+\alpha)} \cdot \frac{(a+\alpha)^1}{1!} + e^{-(a+2\alpha)} \cdot \frac{(a+2\alpha)^2}{2!} + \ldots, \quad (7)$$

and it is valid for all values (real and imaginary) of α when only $|\alpha e^{-\alpha}| < \frac{1}{e}$, and also $|\alpha| < 1$. It is valid, at any rate, for the values of α we are using here, *viz.* the positive numbers between 0 and 1. Just now we shall consider 2 special cases only: $a = 0$ and $a = \alpha$. Then we have

$$\left.\begin{aligned} \sigma_0 &= e^{-0}\frac{0^0}{0!} + e^{-\alpha}\frac{\alpha^1}{1!} + e^{-2\alpha}\frac{(2\alpha)^2}{2!} + \ldots = \frac{1}{1-\alpha} \\ \sigma_{-1} &= e^{-\alpha}\frac{\alpha^0}{0!} + e^{-2\alpha}\frac{(2\alpha)^1}{1!} + \ldots = \frac{1}{1-\alpha}. \end{aligned}\right\} \quad (8\text{—}9)$$

Using (8—9) and (5—6) we find that $a_0 = 1 - \alpha$ which we knew already. Simple expressions can also be found for the quantities s, although not quite so simple as in the case of σ.

6. The application of a_0 to the solution of the main problem.

We will now find $S\binom{>}{z}$, *i. e.* the probability of a waiting time greater than z, or its complement $S\binom{\leq}{z}$. For this purpose, we return to the equation (1) which we shall now proceed to generalize. On the left-hand side we substitute $S\binom{\leq}{z}$ for a_0, and on the right, $s_{(0,-z)}$ for s_0; in other words, we move the oblique line concerned a step z to the left. The equation thus obtained,

$$\left.\begin{aligned} S\binom{\leq}{z} &= 1 - a_0 \cdot s_{(0,-z)} \\ \text{or} \qquad S\binom{>}{z} &= a_0 \cdot s_{(0,-z)} \end{aligned}\right\} \quad (10)$$

is proved in quite the same manner as the original equation (1). Also the equation (2) can be generalized in a similar way, but we need not go into that.

The equation (10) has the drawback of containing an infinite series which, however, can be easily replaced with a finite series. We have

$$r_{(0,-z)} + s_{(0,-z)} = \sigma_{(0,-z)} \tag{11}$$

$$a_0 \cdot \sigma_{(0,-z)} = 1, \tag{12}$$

the latter resulting from *Jensen's* theorem.

By means of this, we get from (10)

$$S\begin{pmatrix}>\\z\end{pmatrix} = 1 - a_0 r_{(0,-z)} \tag{13}$$

or

$$S\begin{pmatrix}<\\z\end{pmatrix} = \qquad a_0 r_{(0,-z)}. \tag{14}$$

This formula is valid for all values of z, but the number of terms resulting depends on whether we are dealing with first interval, $0 < z < t$, or second interval, $t < z < 2\,t$, &c. As I have done elsewhere, certain special constants b_0, b_1; c_0, c_1, c_2, c_3; &c., can here be used to write the formulae concerning each separate interval, but these constants are easily derivable from a_0. As a matter of fact, the formula (14) expresses everything in the simplest and most convenient form.

7. *The case of $x = 2$; definition of a_1 and a_0.*

We understand by a_1 the probability that there will be no waiting time after an arbitrarily chosen call (or that there will be at least one unoccupied line); by a_0 we understand the probability that there will be no waiting time after a call when there has been another call immediately preceding it (or that a random call will find both the lines concerned unoccupied). We get directly the relation,

$$a_1 + a_0 = 2\,(1 - a); \tag{15}$$

for, a_1 is the probability that there will be at least 1 line unoccupied at at any arbitrarily chosen moment, and a_0 is the probability that there will be 2 unoccupied lines; and $2\,(1 - a)$ is the average number of unoccupied lines. — A number of equations sufficient for the determination of a_1 and a_0 will be given later.

8. The table and the oblique lines.

Here, too, we use the previously mentioned table and define certain sums, partly finite, partly infinite, and denoted by the letters σ, r, and s; and we attach the definitions to certain oblique lines having the directional coefficient α and being situated in the plane of the table. The only distinction is that the difference in abscissa for the successive points selected along an oblique line is not 1, but 2; the difference in ordinate is not α, but 2α. As before, we use two indices, *viz.* the abscissa and ordinate for one of the points; the ordinate, however, can be omitted when equal to 0. Additional distinctive marks consisting of 1 vertical stroke, respectively 2 vertical strokes are prefixed in the cases where there is a risk of mistaking the previously defined sums for those now introduced. We have also here

$$r_{(x,\,y)} + s_{(x,\,y)} = \sigma_{(x,\,y)}, \tag{16}$$

where the symbol σ indicates the inclusion of all terms (or all which are not 0); s, on the other hand, indicates the inclusion of those only which correspond to points with positive ordinates; and r, that only those corresponding to negative ordinates, and 0, are included. — (σ and s are equal and $r = 0$ in many cases, *viz.* when the oblique line intersects the negative part of the x axis, or possibly the positive part between the points $x = 0$ and $x = 1$.

9. Determination of a_0 and a_1.

We will prove that

$$\left.\begin{aligned} a_1 &= 1 - (a_1 s_0 + a_0 s_1) \\ a_0 &= 1 - (a_1 s_{-1} + a_0 s_0) \\ 0 &= 1 - (a_1 s_{-2} + a_0 s_{-1}), \end{aligned}\right\} \tag{17—19}$$

where, in accordance with the foregoing,

$$\left.\begin{aligned} s_1 &= e^{-t}\frac{t^3}{3!} + e^{-2t}\frac{(2t)^5}{5!} + \dots \\ s_0 &= e^{-t}\frac{t^2}{2!} + e^{-2t}\frac{(2t)^4}{4!} + \dots \\ s_{-1} &= e^{-t}\frac{t^1}{1!} + e^{-2t}\frac{(2t)^3}{3!} + \dots \\ s_{-2} &= e^{-t}\frac{t^0}{0!} + e^{-2t}\frac{(2t)^2}{2!} + \dots \end{aligned}\right\} \tag{20—23}$$

or, if you like,

$$\left.\begin{aligned} s_1 &= e^{-2\alpha}\frac{(2\alpha)^3}{3!} + e^{-4\alpha}\frac{(4\alpha)^5}{5!} + \dots \\ s_0 &= e^{-2\alpha}\frac{(2\alpha)^2}{2!} + e^{-4\alpha}\frac{(4\alpha)^4}{4!} + \dots \\ s_{-1} &= e^{-2\alpha}\frac{(2\alpha)^1}{1!} + e^{-4\alpha}\frac{(4\alpha)^3}{3!} + \dots \\ s_{-2} &= e^{-2\alpha}\frac{(2\alpha)^0}{0!} + e^{-4\alpha}\frac{(4\alpha)^2}{2!} + \dots \end{aligned}\right\} \qquad (24-27)$$

As we shall see later, the equations (17—19) can be expressed in a different form by introducing the sums σ instead of the sums s; but we will prove them in the form as given above.

The equation (17) can be proved as follows: If we consider all the cases where an arbitrary call must suffer a waiting time, it will be evident that these cases can be arranged in various groups, such as:—

1) During the preceding time interval of the duration t (or 2α) there were 2 or 3 calls
2) - - - - - - - - $2t$ there were 4 or 5 calls
3) - - - - - - - - $3t$ there were 6 or 7 calls, and so on.

Care should be taken, however, that no one case be placed under two different groups; to avoid uncertainty in this respect we will decide upon always preferring the group with the higher number to that with the lower number. Agreement with this is found in that, in group no. 1 above, the specification reads "2 or 3 calls" (*i. e.* and no more), and the following groups are in analogy with this; but the cases which, accordingly, should be included must now be sifted further. It will be necessary to distinguish, within group no. 1, between a subordinate group a (2 calls) and a subordinate group b (3 calls), and similarly within the other groups. It is now easy to see that the probability that a case really belongs under a subordinate group a where it has temporarily been placed, is identical with the probability that an arbitrarily chosen call will not have to suffer a waiting time; in other words, a_1. Likewise, the probability that a case temporarily placed under a subordinate group b really belongs there, is the same as the probability that an arbitrary call will not have to wait and furthermore finds both lines unoccupied; in other words, a_0. Hence we have

$$a_1 = 1 - a_1\left(e^{-t}\frac{t^2}{2!} + e^{-2t}\frac{(2t)^4}{4!} + \ldots\right) - a_0\left(e^{-t}\frac{t^3}{3!} + e^{-2t}\frac{(2t)^5}{5!} + \ldots\right)$$

or

$$a_1 = 1 - (a_1 s_0 + a_0 s_1), \qquad q. e. d.$$

The equations (18) and (19) can be proved in a similar manner, but we shall not dwell on that.

By inserting σ instead of s, the equations (17—19) become simpler and more uniform, *viz.*

$$\left.\begin{aligned} 1 &= a_1\sigma_0 + a_0\sigma_1 \\ 1 &= a_1\sigma_{-1} + a_0\sigma_0 \\ 1 &= a_1\sigma_{-2} + a_0\sigma_{-1} \end{aligned}\right\} \qquad (28—30)$$

The significance of these three equations (their number could easily be increased) is that the constants a_1 and a_0 can be determined by means of any two of them (or by any one of them when the equation (15) is utilized). Incidentally, they are also significant in another respect which we shall see later.

10. Introduction of the new constant β, summation of the infinite series, and determination of a_1 and a_0.

The infinite series, in the summation of which we are now interested, are the following:

$$\left.\begin{aligned} \sigma_1 &= e^0\frac{0^1}{1!} + e^{-2\alpha}\frac{(2\alpha)^3}{3!} + e^{-4\alpha}\frac{(4\alpha)^5}{5!} + \ldots \\ \sigma_0 &= e^0\frac{0^1}{1!} + e^{-2\alpha}\frac{(2\alpha)^2}{2!} + e^{-4\alpha}\frac{(4\alpha)^4}{4!} + \ldots \\ \sigma_{-1} &= e^{-2\alpha}\frac{(2\alpha)^1}{1!} + e^{-4\alpha}\frac{(4\alpha)^3}{3!} + \ldots \\ \sigma_{-2} &= e^{-2\alpha}\frac{(2\alpha)^0}{0!} + e^{-4\alpha}\frac{(4\alpha)^2}{2!} + \ldots \end{aligned}\right\} \qquad (31—34)$$

We know the sums of the following series which are closely related to those just mentioned (*Jensen*'s theorem, equation no. 7 above):

$$
\left.\begin{aligned}
|\sigma_1 &= e^{a}\frac{(-a)^0}{0!} + e^0\frac{0^1}{1!} + e^{-a}\frac{a^2}{2!} + e^{-2a}\frac{(2a)^3}{3!} + \ldots = \frac{1}{1-a}\\
|\sigma_0 &= e^0\frac{0^0}{0!} + e^{-a}\frac{a^1}{1!} + e^{-2a}\frac{(2a)^2}{2!} + e^{-3a}\frac{(3a)^3}{3!} + \ldots = \frac{1}{1-a}\\
|\sigma_{-1} &= e^{-a}\frac{a^0}{0!} + e^{-2a}\frac{(2a)^1}{1!} + e^{-3a}\frac{(3a)^2}{2!} + e^{-4a}\frac{(4a)^3}{3!} + \ldots = \frac{1}{1-a}\\
|\sigma_{-2} &= e^{-2a}\frac{(2a)^0}{0!} + e^{-3a}\frac{(3a)^1}{1!} + e^{-4a}\frac{(4a)^2}{2!} + e^{-5a}\frac{(5a)^3}{3!} + \ldots = \frac{1}{1-a}
\end{aligned}\right\}(35-38)
$$

The series $\|\sigma$ and $|\sigma$ only differ in that every second of the terms contained in the latter is missing in the former. Now, we obtain from the equations (35—38):

$$
\left.\begin{aligned}
\frac{1}{a}\cdot|\sigma_1 &= (ae^{-a})^{-1}\cdot\frac{(-1)^0}{0!} + (ae^{-a})^0\frac{0^1}{1!} + (ae^{-a})^1\frac{1^2}{2!} + \ldots = \frac{1}{a(1-a)}\\
|\sigma_0 &= (ae^{-a})^0\frac{0^0}{0!} + (ae^{-a})^1\frac{1^1}{1!} + (ae^{-a})^2\frac{2^2}{2!} + \ldots = \frac{1}{1-a}\\
a\cdot|\sigma_{-1} &= (ae^{-a})^1\frac{1^0}{0!} + (ae^{-a})^2\frac{2^1}{1!} + (ae^{-a})^3\frac{3^2}{2!} + \ldots = \frac{a}{1-a}\\
a^2\cdot|\sigma_{-2} &= (ae^{-a})^2\frac{2^0}{0!} + (ae^{-a})^3\frac{3^1}{1!} + (ae^{-a})^4\frac{4^2}{2!} + \ldots = \frac{a^2}{1-a}
\end{aligned}\right\}(39-42)
$$

A scheme of obtaining the values of the four quantities

$$\frac{1}{a}\cdot\|\sigma_1,\quad \|\sigma_0,\quad a\cdot\|\sigma_{-1},\quad a^2\cdot\|\sigma_{-2},$$

by removing every second term (*viz.* those with odd exponents) from the four series above, all of which are arranged according to the powers of ae^{-a}, can be put in practice in a convenient way by replacing a in each series with a new constant β, as given by the equation

$$\beta e^{-\beta} = -ae^{-a}$$

and then taking the mean value of the old and the new result. The equation will always have one, and only one, serviceable (negative) root (*i. e.* one to which *Jensen's* theorem can be applied).

Thus, we get

$$\left.\begin{aligned}
\frac{1}{\alpha}\cdot\|\sigma_1 &= \frac{1}{2}\left(\frac{1}{\alpha(1-\alpha)}+\frac{1}{\beta(1-\beta)}\right)\\
\|\sigma_0 &= \frac{1}{2}\left(\frac{1}{1-\alpha}+\frac{1}{1-\beta}\right)\\
\alpha\cdot\|\sigma_{-1} &= \frac{1}{2}\left(\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}\right)\\
\alpha^2\cdot\|\sigma_{-2} &= \frac{1}{2}\left(\frac{\alpha^2}{1-\alpha}+\frac{\beta^2}{1-\beta}\right)
\end{aligned}\right\} \qquad (43-45)$$

or,

$$\left.\begin{aligned}
\|\sigma_1 &= \frac{\alpha}{2}\left(\frac{1}{\alpha(1-\alpha)}+\frac{1}{\beta(1-\beta)}\right)\\
\|\sigma_0 &= \frac{1}{2}\left(\frac{1}{1-\alpha}+\frac{1}{1-\beta}\right)\\
\|\sigma_{-1} &= \frac{1}{2\alpha}\left(\frac{\alpha}{1-\alpha}+\frac{\beta}{1-\beta}\right)\\
\|\sigma_{-2} &= \frac{1}{2\alpha^2}\left(\frac{\alpha^2}{1-\alpha}+\frac{\beta^2}{1-\beta}\right)
\end{aligned}\right\} \qquad (47-50)$$

It is possible, of course, to find the quantities $\| s$ just as we have found the quantities $\| \sigma$ (expressed in terms of α and β), but the expressions will not be quite so neat. — From the equations (28—30) and (47—50) we now obtain

$$\left.\begin{aligned}
a_1 &= 2(1-\alpha)\,\frac{\alpha}{\alpha-\beta}\\
a_0 &= -2(1-\alpha)\,\frac{\beta}{\alpha-\beta}
\end{aligned}\right\} \qquad (51-52)$$

which, by insertion, will satisfy not only (28—30), but also all those analogous to the latter, *i. e.* more generally the equation

$$a_1\,\sigma_p + a_0\,\sigma_{p+1} = 1;$$

for we get

$$2(1-\alpha)\frac{\alpha}{\alpha-\beta}\cdot\frac{\alpha^p}{2}\left(\frac{1}{\alpha^p(1-\alpha)}+\frac{1}{\beta^p(1-\beta)}\right)$$
$$-\,2(1-\alpha)\frac{\beta}{\alpha-\beta}\cdot\frac{\alpha^{p+1}}{2}\left(\frac{1}{\alpha^{p+1}(1-\alpha)}+\frac{1}{\beta^{p+1}(1-\beta)}\right)=1$$

11. Applying the quantities found, a_1 and a_0, to the solution of the main problem.

We shall now determine $S\binom{>}{z}$, *i. e.* the probability of a waiting time greater than z; or its complement $S\left(\frac{\leq}{z}\right)$. We consider the equation (17) which we will now generalize. On the left side we replace a_1 with $S\left(\frac{\leq}{z}\right)$, and on the right s_0 and s_1 with $s_{(0,-z)}$ and $s_{(1,-z)}$, respectively; in other words, we move the oblique line concerned a step z to the left. The equation thus obtained,

$$S\left(\frac{\leq}{z}\right) = 1 - (a_1 \cdot s_{(0,\,-z)} + a_0 \cdot s_{(1,\,-z)}) \tag{53}$$

or

$$S\binom{>}{z} = \quad a_1 \cdot s_{(0,\,-z)} + a_0 \cdot s_{(1,\,-z)} \tag{54}$$

can be proved in quite the same manner as the equation (17). Also (18) and (19) can be generalized in a similar way, but we need not go into that.

Now, the infinite series in the equations (53—54) can be replaced with finite ones. We have

$$r_{(0,\,-z)} + s_{(0,\,-z)} \quad = \sigma_{(0,\,-z)} \tag{55}$$

$$r_{(1,\,-z)} + s_{(1,\,-z)} \quad = \sigma_{(1,\,-z)} \tag{56}$$

$$a_1\sigma_{(0,\,-z)} + a_0\sigma_{(1,\,-z)} = 1; \tag{57}$$

hence

$$S\binom{>}{z} = 1 - a_1 r_{(0,\,-z)} - a_0 r_{(1,\,-z)} \tag{58}$$

or

$$S\left(\frac{\leq}{z}\right) = \quad a_1 r_{(0,\,-z)} + a_0 r_{(1,\,-z)} \tag{59}$$

The formula is valid for all values of z. The number of terms contained in the formula depends on whether z belongs in the first interval $0 < z < t$, or in the second interval $t < z < 2\,t$, and so on.

It is easy to write out, as I have done elsewhere, the special formulae valid for the separate intervals; the constants involved here, b_0, b_1, b_2, b_3; $c_0, c_1, c_2, c_3, c_4, c_5$; &c., are easily derived from a_0 and a_1. However, the formula (58—59) really expresses everything, and perhaps even in the very best form, at that.

12. Appendix.

The proof of the theorem used in the above, *viz.:* When, during a given time, the average number of calls is y, the probability of x calls being originated will be

$$S_x = e^{-y} \frac{y^x}{x!}.$$

Let it be assumed that the time in consideration represents a portion of a very long time over which a correspondingly great number of calls is dispersed so that y calls, at an average, fall within the time portion considered. The duration of the latter can be called y, the unit of time being chosen in such a manner as to give an average of 1 call per unit of time. Let us suppose that, in a certain case, say, 5 calls occur within the time y, and let us move y a short distance dy; then there will be a probability $\frac{5\,dy}{y}$ that 1 of the 5 calls is shut out so that the number is reduced to 4. *Vice versa,* if we had 4 calls before y was moved, there will be a probability $\frac{5\,dy}{y}$ of gaining 1 new call by the movement. But the transitions from 5 to 4 and *vice versa* must neutralize each other, and so

$$S_5 \cdot \frac{5}{y} = S_4.$$

This result — and analogous results — give us the ratio between the successive members of the sequence $S_0, S_1, S_2, \ldots$; these must then be proportional to

$$1, \quad \frac{y}{1!}, \quad \frac{y^2}{2!}, \quad \frac{y^3}{3!}, \ldots$$

As, necessarily,

$$S_0 + S_1 + S_2 + \ldots = 1,$$

and as

$$1 + \frac{y}{1!} + \frac{y^2}{2!} + \frac{y^3}{3!} + \ldots = e^y,$$

we obtain

$$S_0 = e^{-y}, \quad S_1 = e^{-y} \frac{y}{1!}, \quad S_2 = e^{-y} \frac{y^2}{2!} \ldots,$$

q. e. d.

Table 1.

Values of the constants a_0 for $x = 1$; a_1 and a_0 for $x = 2$.

	$x = 1$	$x = 2$	
α	a_0	a_1	a_0
0·0	1·0	1·000000	1·000000
1	0·9	0·982234	0·817766
2	8	935507	664493
3	7	866418	533582
4	6	779180	420820
5	5	676741	323259
6	4	561316	238684
7	3	434633	165367
8	2	298104	101896
9	1	152892	047108
1·0	0	000000	000000

Table 2.

Values of $S\left(\frac{<}{z}\right)$ for $x = 1$ and $x = 2$.

$x = 1$.

α \ z	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
0·0	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
1	0·900	0·995	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
2	800	884	0·977	0·991	0·998	0·999	1·000	1·000	1·000	1·000	1·000
3	700	774	855	945	967	983	0·992	0·996	0·998	0·999	0·999
4	600	663	733	810	895	923	947	965	977	984	990
5	500	553	611	675	746	824	856	885	910	931	947
6	400	442	489	540	597	659	729	761	792	822	849
7	300	332	366	405	448	495	547	605	635	665	694
8	200	221	244	270	298	330	364	403	445	470	495
9	100	111	122	135	149	165	182	201	223	246	261
1·0	000	000	000	000	000	000	000	000	000	000	000

$x = 2.$

a \ z	·0	·1	·2	·3	·4	·5	·6	·7	·8	·9	1·0
0·0	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
1	0·982	0·995	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000	1·000
2	936	960	0·980	0·994	0·999	1·000	1·000	1·000	1·000	1·000	1·000
3	866	899	928	953	974	0·989	0·995	0·997	0·999	0·999	1·000
4	779	815	849	881	911	938	960	976	985	989	0·992
5	677	712	748	783	817	849	880	909	931	949	961
6	561	594	627	661	695	729	762	794	824	852	877
7	435	462	490	520	550	580	611	642	673	703	732
8	298	318	339	361	384	407	432	457	482	508	533
9	153	164	175	187	200	213	227	241	256	272	288
1·0	000	000	000	000	000	000	000	000	000	000	000