Fall 1999 Course 513-697: Applied Linear Models

Highlights / Key Concepts in NKNW4 Chapter 7

pages 8-10

Ideas unique to Multiple Regression - no analogy with Simple Linear Regression.
A number of important sub-topics are introduced in this Chapter (*see also* "*Chapter 9*" ...

--Formal tests for Addition/Deletion of 1 or more terms in a regression model (nicely do...

-- Collinearity -- very important -- should be in a chapter by itself!

-- Interaction terms -- again, should be in a chapter by itself ("Effect Modification" in Epidemiology)!

7.1 Extra Sums of Squares

Extra SS if focus on SS_{reg}: *Reduced* SS if focus on $SS_{residual}$:

Same idea whether adding or removing 1 (or more) term(s)

EXAMPLE

Y: % Body Fat (by underwater weighing !!)

X_1 Skinfold Thickness (by calipers:- some discomfort)

X_2 Skinfold Thickness (by tape measure:- painless)

X_3 Midarm Circumference (by tape measure:- painless)

*IN COMPACT NOTATION (**2 "X" terms**)*

SSR (short for SS_{Reg})		**SSE** (short for SS_{Error})		SST (short for SS_{total})
$SSR(X_1)$	+	$SSE(X_1)$	=	SST
$SSR(X_1, X_2)$	+	$SSE(X_1, X_2)$	=	SST
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$SSR(X_1, X_2) - SSR(X_1)$	=	$SSE(X_1) - SSE(X_1, X_2)$		
"Extra SSR due to X_2"	=	"Reduced SSE due to X_2"		
$SSR(X_2 \mid X_1)$				

*IN COMPACT NOTATION (**3 "X" 's**)*

$SSR(X_1)$	+	$SSE(X_1)$	=	SST
$SSR(X_1, X_2)$	+	$SSE(X_1, X_1)$	=	SST
$SSR(X_1, X_2, X_3)$	+	$SSE(X_1, X_1, X_3)$	=	SST

=>	$SSR(X_1)$	= SST	-	$SSE(X_1)$	*Note*: **order matters**!
	$SSR(X_2 \mid X_1)$	= $SSE(X_1)$	-	$SSE(X_1, X_2)$	$SSR(X_2 \mid X_1)$ not same
	$SSR(X_3 \mid X_1, X_2)$	= $SSE(X_1, X_2)$	-	$SSE(X_1, X_2, X_3)$	as $SSR(X_1 \mid X_2)$

Different decompositions have different meanings

e.g.	a	b	c
	$SSR(X_1)$	$SSR(X_1)$	$SSR(X_3)$
	$SSR(X_2 \mid X_1)$	$SSR(X_3 \mid X_1)$	$SSR(X_2 \mid X_3)$
	$SSR(X_3 \mid X_1, X_2)$	$SSR(X_2 \mid X_1, X_3)$	$SSR(X_1 \mid X_2, X_3)$

*These decompositions are referred to as "**variables added in order** SS" or "**Type I** SS"*
*(to distinguish them from "**variables added last** SS" or "**Type III** SS" used below)*

If labeling is not obvious, or if you forget what Type I means (I don't blame you, given the imaginative choice of the label), one way to recognize it for what it is is that Type I SS count each separate contribution just once .. so that the successive Type I SSR's add to the overall SSR.

Graphical Depiction of these decompositions ... see Figure 7.1 on page 265

(I find this diagram so helpful that I scanned it and put it on the www page for c678 -- where we used a different text)

Using SS decomposition in ANOVA Table => Decomposition of Degrees of Freedom

The above decomposition is for the Sums of Squares.

When the decomposition is done one term at a time, the $SSR(X_1)$, $SSR(X_2 \mid X_1)$, $SSR(X_3 \mid X_1, X_2)$ ... and the corresponding $MS(X_1)$, $MS(X_2 \mid X_1)$, $MS(X_3 \mid X_1, X_2)$ are the same -- since each divisor involves 1 df.

7.2
and
7.3 **"Extra Sums of Squares" to test coefficients of Multiple Regression**

From Cochran's Theorem [partitioning SS; each $(1/\sigma^2)$ MS ~ indep. central/non-central X^2]

Single β [H_0: **this** β's = 0]

t* = b / SE[b] with df = df for residuals (e's) when this X & other X's in the model

or (***equivalently***, since $t^{*2} = F^*$)

$$F^* = \frac{\text{SSR[this X | other X's] / 1 df}}{\text{SSE[this X \& other X's] / df for e's if this X \& other X's}} = \frac{\text{MSR[this X | otherX's]}}{\text{MSE[this X \& other X's]}}$$

Several β's [H_0: **these** β's = 0]

$$F^* = \frac{\text{SSR[these X's | other X's] / \# of these X's}}{\text{SSE[these \& other X's] / df for E if these X's \& other X's}} = \frac{\text{MSR[these X's | other X's]}}{\text{MSE[these \& other X's]}}$$

All β's [H_0: **all the** β's = 0]

$$F^* = \frac{\text{SSR[all these X's] / \# of X's}}{\text{SSE[these X's] / df for e's if these X's}} = \frac{\text{MSR[these X's]}}{\text{MSE[these X's]}}$$

IN GENERAL [H_0: **set of (linear) constraints on the** β's]

$$F^* = \frac{\text{SSR[constrained } \beta\text{'s] / \# of constraints}}{\text{SSE[no constraints] / df for e's if no constraints}} = \frac{\text{MSR["due to" constrained } \beta\text{'s]}}{\text{MSE[no constraints]}}$$

i.e.

$$F^* = \frac{\{SS_{Reg}\text{[larger model]} - SS_{Reg}\text{[smaller model]}\} \text{ / \# of constraints}}{\text{SSE[no constraints] / df for e's in larger model}}$$

$$= \frac{\text{MSR["due to" constrained } \beta\text{'s]}}{\text{MSE[larger model]}}$$

Q: Why use the MSE from the larger model as the denominator of each test?

Even if some terms in larger model are unnecessary, this MSE is an unbiased estimator of $\sigma^2[\varepsilon]$.

7.4 Coefficients of Partial Determination

***Recall*:** Coefficient of Multiple Determination:

Reduction in Var(Y) through use of X_1, X_2, ..

$$r^2_{\text{Y with X1, X2, X3, ...}} = \frac{\text{Var}[Y] - \text{Var}[Y \mid X_1\ X_2\ ...]}{\text{Var}[Y]} = \frac{\text{SSR}[X_1\ X_2\ ...]}{\text{SSE[model with just } b_0]}$$

By analogy: Coefficient of Partial Determination (by X_1 -- after already fitting X_2):

Reduction in Var(residual $Y \mid X_2$)

$$r^2_{\text{Y with X1 given X2}} = \frac{\text{Var}[Y \mid X_2] - \text{Var}[Y \mid X_1\ X_2]}{\text{Var}[Y \mid X_2]}$$

$$= \frac{\text{SSR}[X_1 \mid X_2]}{\text{SSE[model with just } \{X_0 \text{ and}\}\ X_1]}$$

likewise...

$$r^2_{\text{Y with X2 given X1}} = \frac{\text{Var}[Y \mid X_1] - \text{Var}[Y \mid X_1\ X_2]}{\text{Var}[Y \mid X_1]}$$

$$= \frac{\text{SSR}[X_2 \mid X_1]}{\text{SSE[model with just } \{X_0 \text{ and}\}\ X_2]}$$

note:

$r^2_{\text{Y with X1 given X2}} = r^2_{\text{Y' with X1'}}$,

where Y' = residual of Y after regressing Y on X_2, and

X_1' = residual of X_1 after regressing X_1 on X_2 .

Going on to more X's ...

$$r^2_{\text{Y with X4 given X1 , X2 and X3}} = \frac{\text{SSR}[X_4 \mid X_1\ X_2\ X_3]}{\text{SSE[model with } X_1\ X_2\ X_3]}$$

Coefficients of Partial Correlation (Square Root of Coefficient of Partial determination)

(Useful to those who think in correlations rather than regresssion coefficients)

$r_{\text{Y with X4 given X1 , X2 and X3}}$

$$= \text{sign}[b_4 \text{ in model of Y on } X_1 \text{ to } X_4] \times \sqrt{r^2 \text{ of Y with } X_4 \text{ given } X_1 \text{ to } X_3}$$

(see eqns. 7.41 & 7.42 on how to calculate them from lower order partial correlation coefficients)

7.5 Standardized Multiple Regression Model

- relevant also for Simple Regression model:- eliminates b's in different Y/X_1 , Y/X_2,... units (so don't have to worry if X_1 is in cm or inches or metres, X_2 in Kg or lbs or grams-- or $'s)

- issues of numerical accuracy no longer quite as relevant with high-precision facilities (accuracy issues if X's of very different magnitudes, or highly correlated so det[$\mathbf{X}^T\mathbf{X}$] close to 0)

Correlation Transformation

- transform each X variable so that each entry in $\mathbf{X}^T\mathbf{X}$ matrix is a correlation of 2 X's , i.e.

$$X' = \frac{X' - Xbar}{\sqrt{\Sigma(X - Xbar)^2}}$$

... $X'^T X'$ matrix = correlations of pairs of X's

(text formula makes transf. unnec. complex)

- transform Y variable in same way , i.e.

$$Y' = \frac{Y' - Ybar}{\sqrt{\Sigma(Y - Ybar)^2}}$$

(text formula makes transf. unnec. complex)

- fit model $Y' = \beta_1' X_1' + \beta_2' X_2' + ... + e'$ (*no intercept* since ave[Y '] now = 0)

work back ...

$$\beta_k = \beta_k' \frac{SD[Y]}{SD[X_k]}$$

$$\beta_0 = Ybar - (\beta_1 X_1 + \beta_2 X_2 + ...)$$

$X'^T X'$ matrix = correlation matrix of original X's = R_{XX}

$X'^T Y'$ vector = correlation of Y with each original X = R_{YX}

LS Solution for **b '** , the column vector $(\beta_1', \beta_2', ...)^T$

$$\mathbf{b}' = (R_{XX})^{-1} R_{YX}$$

For worked example, see text

7.6 Multicollinearity and its Effects (see also item from Graybill on Ill-Conditioning)

Meaning: (high) intercorrelation among some or all of the X terms in a multiple regression

Implications:

- easiest to understand by examining just 2 Xs and just the two extremes:

- see "Confounding[in pictures and numbers]" in Chapter 8 material in c678 www page;
- see also the "hammock" spreadsheet

X1 & X2 uncorrelated	X1 & X2 perfectly (+ or -) correlated
• b_1 (X1-only model) = b_1 (X1 & X2 model)	b_1 (X1 only model) doesn't have same meaning or value as b_1 (X1&X2 model)
• SSR(X1) = SSR(X1 | X2)	SSR(X1 | X2) = SSR(X2 | X1) = 0
and vice versa (can fit b1 and b2 "marginally")	different (b_1,b_2) pairs give same fit (see Fig 7.2 in NWNW4)
uncorrelated estimates of β_1 and β_2	cannot "separate" β_1 and β_2 estimates:- (b_1,b_2) are unstable • (b_1+ve , b_2 -ve) in one sample • (b_1-ve , b_2 +ve) in another
	Like hammock, fitted surface rests on a "knife-edge"
	BUT: can make predictions within (X1,X2) data region

In practice, inter-correlations of X's (and effects of these) are usually somewhere in between. but, more difficult to "see" if more than 2 X's. ***(Large SE's for b's can be a warning)***

Chapter 9 will describe methods for detecting multicollinearity in "higher-D" X data.
Chapter 10 explains remedial methods (including "Ridge Regression").

7.7 Polynomial Regression

(Use of higher powers of X (or products of two or more X's) as terms in a multiple regression) Can be helpful for fitting Response Functions of a single X, or a Response Surface for 2 X's.

• INSIGHT (in SAS) has a dangerously-simple interface for fitting a polynomial in a single X.

• Polynomial regression is simply a multiple regression where the terms are powers of same basic X variable; BUT one needs to be extra careful about overfitting and about extrapolation This will remind me to relate one physician's use of a fitted polynomial of time (fitted in the original Lotus1-2-3 software) to monitor (and ?? anticipate) a patient's White Blood Cell (WBC) count over time.

• *On a related note*: concerning Fig1 of the article " ***Changes in Alcohol Consumption With Age***" in Can J Public Health Vol. 82, July/August 1991 pp231-4 elsewhere on this www page. (see also excerpts from article)

RESULTS (from text of the article)

Measures of consumption (based on a total of 3,304 interviews)

The top panel of Figure 1 shows mean alcohol consumption in drinks per month for males, females and all respondents by age. The middle panel presents in a similar manner the frequency of drinking occasions per month, and the bottom panel shows the quantity (mean number of drinks consumed per drinking occasion).

These figures were derived by rank ordering all respondents by age. ***Each data point represents the mean of successively older groups of 25 respondents****. Plotting data in this fashion provides information on the relative density of observations according to age and sex.* ***In order to reduce the scatter which could obscure trends, a 4253H, twice compound smoother with endpoint adjustment was used. This consists of a series of running median smoothers and the Hanning running weighted average smoother applied twice****.*

The top panel shows an age-related decline in total alcohol consumption per month for all respondents. This line has a slope of -0.12. It is clear from these figures that the age-related decrease seen in the top panel is largely due to a decrease in quantity which shows a rather steady decline with age (slope = -0.26), rather than any change in frequency, which has a slope of only -.04. The correlation between frequency and dose is -0.11 which is small, but statistically significant (p < 0.0001).

• I question the choice of "smoothing" that the authors carried out. Although they are non-parametric, *they look like high order polynomials that seem to "follow" every little random twist and turn in the raw data*. To my eye, the patterns in the bottom panel are quite linear--- the "join the dots" approach (even with each dot being a running median) over-accentuates the random components -- what one wants first is the BIG PICTURE .. the clear downwards "close to linear" trend. I believe the little ups and downs along the way are random noise -- and that they are being over-emphasized. I cannot imagine that the population medians actually behave like this.

• Polynomial models are often used for prediction. Thus, the meaning of individual β's is less critical than when they are associated with different X's. Nevertheless, they are a good example of **the benefits of "centering" X variables in any multiple regression**, and of the **induced collinearity if one uses "raw" powers of X, or -- for that matter --products of two different X variables**. If X is a positive RV, then X^2 can be strongly correlated with it.

7.8 Regression Models involving Interactions

See also: www material, Session 5, course 678: Interaction(Effect Modification) in Regression

Definitions ...

Interaction *(statistical)*
- "Non-additivity" of "effects" in regression
- need for product term in regression analysis (miettinen)
  note that need for product term may be scale-dependent (e.g. Y vs log Y scale)

(Effect) Modification *(epidemiological)*
- Inconstancy of a parameter of a relation over other subject characteristic (miettinen)
- "Different slopes for different folks" (jh)

"***Modifier*** (of a relation)
- A characteristic (of individuals) on which a parameter of a relation depends (osm)

Examples... *(first 4 are on course 678 www page)*

Equation for Ideal Weight as function of Height	- modification by Gender
Average Earnings as function of Education / Age	- modification by Gender
Decline in Bone Density with Age	- Different in 19th and 20th Centuries
Hit further with aluminum than wood baseball bat?	- Depends on *where on bat* one hits
Changes over time in injury rates	- Different in intervn. & ref. areas?

Comments on NKNW4 (and most statisticians') terminology:

"regression model is not additive, or. equivalently, it contains an ***interaction effect*** "

"non-additive" is a more informative phrase; it avoids possible over-interpretation of the word interaction; it highlights the non constancy (over X_2) of effect of X_1 on (a specific function of) μ_Y. It also plays down the interpretation of phrases like "... interaction effect is of a *reinforcement or synergistic* type" or "of an *interference* or *antagonistic* type". (p311)

Oxford English Dictionary ...

interact:- "to act reciprocally, to act on each other"

interaction:- "reciprocal action, action of persons or things on each other"

Interpretation of models that include interaction (product) terms

$E[Y \mid X_1, X_2] = \beta_0 + \beta_1 X_1 + \beta_2 X_2 + \beta_3 X_1 X_2$

$= (\beta_0 + \beta_2 X_2) + (\beta_1 + \beta_3 X_2) X_1$ ... if wish to think of E[Y] vs X_1

$= (\beta_0 + \beta_1 X_1) + (\beta_2 + \beta_3 X_1) X_2$ ... if wish to think of E[Y] vs X_2

mathematically symmetric in X_1 and X_2 (although seldom so in practice)

Interpretation ...

If one of the two X's is the natural "modifier" of the "Y - other X" relation, easier to refer to modifier by another symbol (M) and write equation as

$$E[Y \mid X, M] = \beta_0 + \beta_1 X + \beta_2 M + \beta_3 X M$$
$$= (\beta_0 + \beta_2 M) + (\beta_1 + \beta_3 M) X$$

Don't try to interpret the β_3 "in isolation". And, although the several different (M-specific) X-Y relations can be represented in a single equation, remember that they must be separated out when describing them.. i.e.

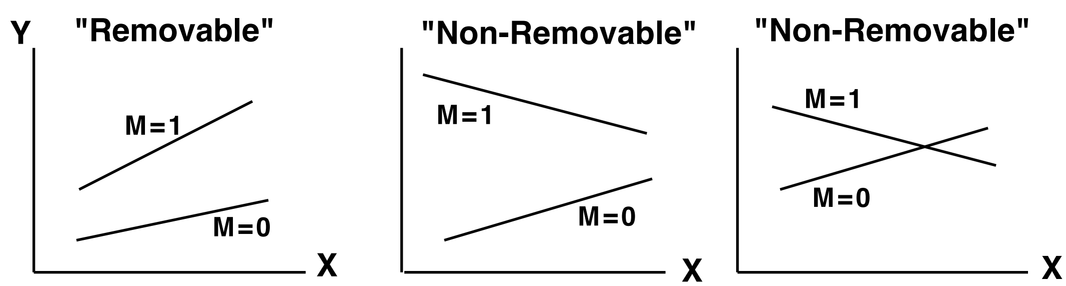
interaction <---> "Different Y vs X story" for each level of M
"no single summary that applies to all levels of M"

Think of β for product term as "additional Y-X slope for each unit difference in M"

If β for product term is small (in sense that Y-X slope isn't that different from one end of the M range to the other) , then the Y-X slope obtained by dropping M from model is not that misleading

("105 lbs. + 5.5 lbs. for every inch over 5 feet" ... for adults of either sex??)

"Removable" vs. "Non-Removable" Interactions



X & M both Binary => β associated with XM product is a"double difference"

$$[\ \bar{y}_{X=1} - \bar{y}_{X=0}\ ]_{M=1} - [\ \bar{y}_{X=1} - \bar{y}_{X=0}\ ]_{M=0}$$

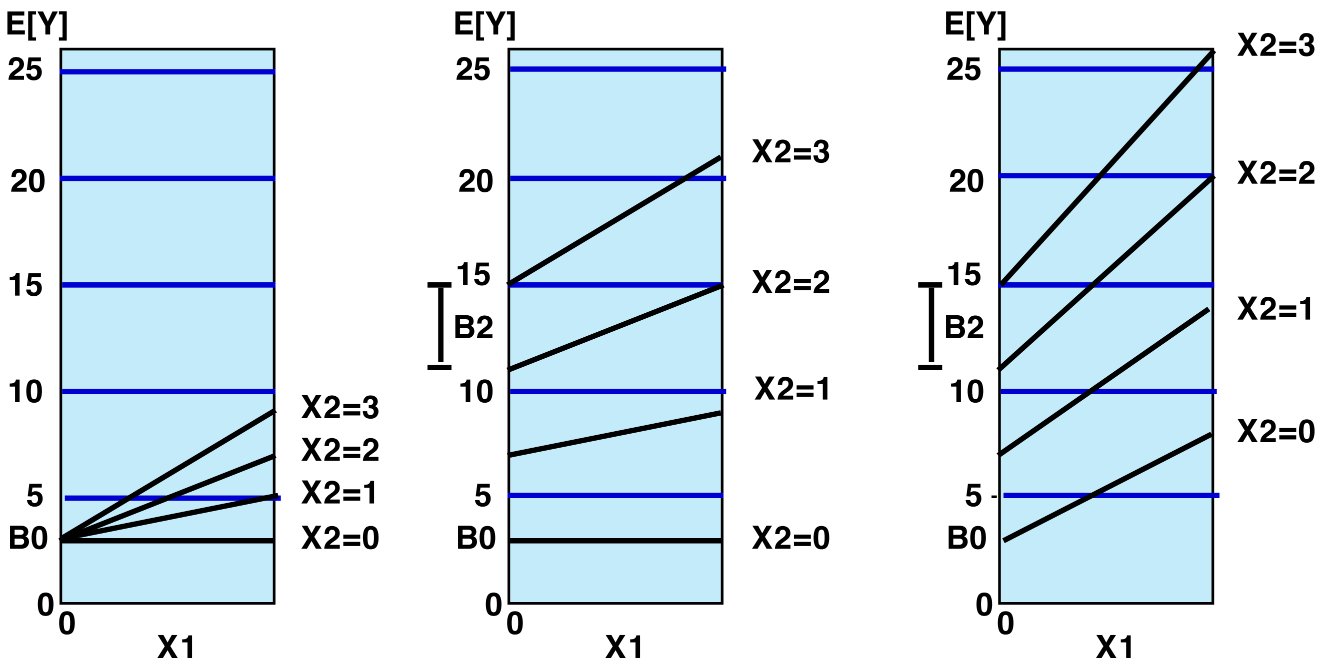
Statistical precision/power to measure/detect a double difference considerable lower than that to measure a single difference. We often end up not being able to adequately statistically test if X differences in response are M-specific, and so depend on analogy or other outside information judgment when deciding whether to report them separately (M-specifically).

Warning in most textbooks: if put product term in model, then must also include each component of the product (i.e. X_1 and X_2 as well as X_1X_2)

Absolutely must? No. BUT be careful to interpret coefficients carefully! Helps to draw the lines e.g.

$E[Y \mid X_1, X_2] =$

$B_0 + B_3X_1X_2$ $\quad\quad$ $B_0 + B_2X_2 + B_3X_1X_2$ $\quad\quad$ $B_0 + B_1X_1 + B_2X_2 + B_3X_1X_2$



Product Terms to test change in Y level or Y-time slope (or both) when changes introduced (serially)

Examples

- Prescriptions filled before and after the introduction of the Quebec Drug Plan
- Motor vehicle fatality rates before/after change back to 65 mph limits
- Asthma deaths before/after removal of certain asthma drug in New Zealand
- Numbers of Marriage Licences issued before/after HIV tests became mandatory
- What Does It Take to Heat a New Room? (see datasets on 697 www page)

Product Terms to test change in Y level or Y-time slope (or both) when changes introduced (parallel groups)

Example

-The Lidkoping Accident Prevention Programme -- a community approach to preventing childhood injuries in Sweden (see www material in "datasets" in course 626)

Reducing Collinearity of Product Term and its Components ... by Centering

Example

-The Lidkoping Accident Prevention Programme (X=Time M = Program)

7.9 Constrained regression

- See text