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A note on the sum of uniform random variables

Aniello Buonocore [a,*], Enrica Pirozzi [a], Luigia Caputo [b]

[a] Dipartimento di Matematica e Applicazioni, Universitá di Napoli Federico II, Via Cintia, 80126 Napoli, Italy

[b] Dipartimento di Matematica, Universitá di Torino, Via Carlo Alberto 10, 10123 Torino, Italy

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ABSTRACT

An inductive procedure is used to obtain distributions and probability densities for the sum S_n of independent, non-equally uniform random variables. Some known results are then shown to follow immediately as special cases. Under the assumption of equally uniform random variables some new formulas are obtained for probabilities and means related to S_n. Finally, some new recursive formulas involving distributions are derived.

1. Introduction

The problem of calculating the distribution of the sum S_n of n uniform random variables has been the object of considerable attention even in recent times. The motivation can be ascribed to various reasons such as the necessity of handling data drawn from measurements characterized by different levels of precision (Bradley and Gupta, 2002), or questions appearing in change point analysis (Sadooghi-Alvandi et al., 2009), or, more in general, the need of aggregating scaled values with differing numbers of significant figures (Potuschak and Müller, 2009). It appears that this problem has been taken up first in Olds (1952), where by somewhat obscure procedures formulas for the probability density function of S_n and its distribution function are derived. An accurate bibliography of articles published in the last century is found in Bradley and Gupta (2002), where the authors also obtain the probability density function of S_n by non-probabilistic arguments, namely via a complicated analytical inversion of the characteristic function. Such a procedure was successively and successfully simplified in Potuschak and Müller (2009), where again no trace of probabilistic arguments is present. An attempt to achieve the same results by a simpler procedure appears in Sadooghi-Alvandi et al. (2009) where a given function is assumed to be the unknown probability density function, the proof of the correctness of such an ansatz being that its Laplace transform coincides with the moment generating function of S_n. Quite differently, the present note includes a novel proof of the above cited results (Proposition 2.1). This is based on an inductive procedure, suitably adapted to our general instance, used by Feller (1966) for the case of identically distributed variables, that further pinpoints the usefulness of induction procedures in the probability context. (See also Hardy et al. (1978) for some more illuminating examples.) In the case of identically distributed random variables, some results concerning certain probabilities and means of random variables related to S_n are obtained (Lemma 3.1, Theorem 3.1, Corollaries 3.1 and 3.2, Proposition 3.4), as well as certain recurrence relations that are reminiscent of those holding for Stirling numbers (Propositions 3.5–3.7).

2. The general case

Let $\{X_n\}_{n\in\mathbb{N}}$ denote a sequence of uniform distributed independent random variables and denote $S_n = \sum_{i=1}^{n} X_i$. Without loss of generality we assume that $X_n \sim U(0, a_n)$ with a_n positive real numbers. By adopting a suitably modified procedure

* Corresponding author. Tel.: +39 081 675684; fax: +39 081 675665.
E-mail addresses: aniello.buonocore@unina.it (A. Buonocore), enrica.pirozzi@unina.it (E. Pirozzi), luigia.caputo@unito.it (L. Caputo).

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due to Feller (1966) we shall obtain the probability density function $f_n(x)$ and the distribution function $F_n(x)$ of S_n for all $n \in \mathbb{N}$. The starting point is to write

$$F_{X_n}(x) = \frac{x^+ - (x - a_n)^+}{a_n}, \quad \forall n \in \mathbb{N}, \ \forall x \in \mathbb{R}, \tag{1}$$

where $(x - c)^+ = \max\{x - c, 0\}, \forall c \in \mathbb{R}$. Next we shall make use of

$$\int_{-\infty}^{x} \left[(y - c)^+\right]^{n-1} \mathrm{d}y = \frac{1}{n} \left[(x - c)^+\right]^n, \quad \forall n \in \mathbb{N}, \forall c \in \mathbb{R}^+. \tag{2}$$

In addition we note that, by convolution, probability density functions and distribution functions are related as follows:

$$f_{n+1}(x) = \int_0^{a_{n+1}} f_n(x - y) f_{X_{n+1}}(y) \, \mathrm{d}y = \frac{F_n(x) - F_n(x - a_{n+1})}{a_{n+1}}, \quad \forall n \in \mathbb{N}, \forall x \in \mathbb{R}. \tag{3}$$

Claim 2.1. *One has*

$$F_1(x) = \frac{x^+ - (x - a_1)^+}{a_1}, \quad \forall x \in \mathbb{R} \tag{4}$$

and

$$f_2(x) = \frac{x^+ - (x - a_1)^+ - (x - a_2)^+ + [x - (a_1 + a_2)]^+}{a_1 a_2}, \quad \forall x \in \mathbb{R}. \tag{5}$$

Proof. It follows from (1) written for $S_1 \equiv X_1$, and from (3). □

Claim 2.2. *One has*

$$F_2(x) = \frac{(x^+)^2 - [(x - a_1)^+]^2 - [(x - a_2)^+]^2 + \left\{[x - (a_1 + a_2)]^+\right\}^2}{2 a_1 a_2}, \quad \forall x \in \mathbb{R} \tag{6}$$

and

$$\begin{aligned} f_3(x) = &\Big\{(x^+)^2 - \left[(x - a_1)^+\right]^2 - \left[(x - a_2)^+\right]^2 - \left[(x - a_3)^+\right]^2 \\ &+ \left\{[x - (a_1 + a_2)]^+\right\}^2 + \left\{[x - (a_1 + a_3)]^+\right\}^2 + \left\{[x - (a_2 + a_3)]^+\right\}^2 \\ &- \left\{[x - (a_1 + a_2 + a_3)]^+\right\}^2\Big\} (2 a_1 a_2 a_3)^{-1}, \quad \forall x \in \mathbb{R}. \end{aligned} \tag{7}$$

Proof. Eq. (6) follows from (5) and (2). From (6) and (3) one then obtains Eq. (7). □

Claims 2.1 and 2.2 lead us to infer a possible general forms of the distribution function of S_n and of the probability density function of S_{n+1}, as specified in the following proposition.

Proposition 2.1. *The distribution function $F_n(x)$ of S_n and the probability density function $f_{n+1}(x)$ of S_{n+1} are given by, respectively:*

$$F_n(x) = \frac{1}{n!\, A_n} \left\{ (x^+)^n + \sum_{\nu=1}^{n} (-1)^\nu \sum_{j_1=1}^{n} \sum_{j_2=j_1+1}^{n} \cdots \sum_{j_\nu=j_{\nu-1}+1}^{n} \left\{ \left[x - (a_{j_1} + a_{j_2} + \cdots + a_{j_\nu})\right]^+ \right\}^n \right\}, \quad \forall n \in \mathbb{N}, \forall x \in \mathbb{R} \tag{8}$$

and

$$f_{n+1}(x) = \frac{1}{n!\, A_{n+1}} \left\{ (x^+)^n + \sum_{\nu=1}^{n+1} (-1)^\nu \sum_{j_1=1}^{n+1} \sum_{j_2=j_1+1}^{n+1} \cdots \sum_{j_\nu=j_{\nu-1}+1}^{n+1} \left\{ \left[x - (a_{j_1} + a_{j_2} + \cdots + a_{j_\nu})\right]^+ \right\}^n \right\}, \quad \forall n \in \mathbb{N}, \forall x \in \mathbb{R}. \tag{9}$$

Proof. We proceed by induction. Claims 2.1 and 2.2 show that Eqs. (8) and (9) hold for $n = 1$ and $n = 2$. Let us now assume that they hold for $n = r - 1$ and prove that they also hold for $n = r$. To this purpose, we re-write Eq. (9) for $n = r - 1$ and $x = y$ and then integrate both sides over $(-\infty, x)$. By virtue of Eq. (2), Eq. (8) with $n = r$ then follows. To obtain Eq. (9) for

$n = r$ we make use of (3) and of the just obtained expression of $F_r(x)$. Hence,

$$f_{r+1}(x) = \frac{1}{r!\,A_{r+1}}\left\{(x^+)^r + \sum_{\nu=1}^{r}(-1)^\nu \sum_{j_1=1}^{r}\sum_{j_2=j_1+1}^{r}\cdots\sum_{j_\nu=j_{\nu-1}+1}^{r}\left\{\left[x-(a_{j_1}+a_{j_2}+\cdots+a_{j_\nu})\right]^+\right\}^r + -\left[(x-a_{r+1})^+\right]^r + \right.$$
$$\left. - \sum_{\nu=1}^{r}(-1)^\nu \sum_{j_1=1}^{r}\sum_{j_2=j_1+1}^{r}\cdots\sum_{j_\nu=j_{\nu-1}+1}^{r}\left\{\left[x-(a_{j_1}+a_{j_2}+\cdots+a_{j_\nu}+a_{r+1})\right]^+\right\}^r\right\}. \tag{10}$$

Eq. (10) identifies with Eq. (9) written for $n = r$ since the curly brackets contains all and only all the following terms:

1. $[(x)^+]^r$;
2. $\left[(x-a_1)^+\right]^r, \left[(x-a_2)^+\right]^r, \ldots, \left[(x-a_{r+1})^+\right]^r$;
3. for $1 < \nu \le r$

$$(-1)^\nu \sum_{j_1=1}^{r}\sum_{j_2=j_1+1}^{r}\cdots\sum_{j_\nu=j_{\nu-1}+1}^{r}\left\{\left[x-(a_{j_1}+a_{j_2}+\cdots+a_{j_\nu})\right]^+\right\}^r +$$
$$-(-1)^{\nu-1} \sum_{j_1=1}^{r}\sum_{j_2=j_1+1}^{r}\cdots\sum_{j_{\nu-1}=j_{\nu-2}+1}^{r}\left\{\left[x-(a_{j_1}+a_{j_2}+\cdots+a_{j_{\nu-1}}+a_{r+1})\right]^+\right\}^r$$
$$\equiv (-1)^\nu \sum_{j_1=1}^{r+1}\sum_{j_2=j_1+1}^{r+1}\cdots\sum_{j_\nu=j_{\nu-1}+1}^{r+1}\left\{\left[x-(a_{j_1}+a_{j_2}+\cdots+a_{j_\nu})\right]^+\right\}^r;$$

4. $(-1)^{r+1}\left\{[x-(a_1+a_2+\cdots+a_{r+1})]^+\right\}^r$.

This complete the induction. □

3. A special case

Let us assume that the random variables in $\{X_n\}_{n\in\mathbb{N}}$ are identically distributed.

Proposition 3.1. *When $a_n = a > 0$ for all $n \in \mathbb{N}$ then*

$$F_n(x) = \frac{1}{n!\,a^n}\sum_{\nu=0}^{n}(-1)^\nu \binom{n}{\nu}\left[(x-\nu a)^+\right]^n, \quad \forall n \in \mathbb{N}, \forall x \in \mathbb{R} \tag{11}$$

and

$$f_{n+1}(x) = \frac{1}{n!\,a^{n+1}}\sum_{\nu=0}^{n+1}(-1)^\nu \binom{n+1}{\nu}\left[(x-\nu a)^+\right]^n, \quad \forall n \in \mathbb{N}, \forall x \in \mathbb{R}. \tag{12}$$

Proof. Eq. (11) follows from Eq. (8) after noting that now $A_n = a^n$ and that

$$a_{j_1} + a_{j_2} + \cdots + a_{j_\nu} = \nu a$$

for $\nu = 0, 1, \ldots, n$. Indeed, in the sum on ν in Eq. (8), the term in curly bracket becomes $\left[(x-\nu a)^+\right]^n$, so that

$$\sum_{j_1=1}^{n}\sum_{j_2=j_1+1}^{n}\cdots\sum_{j_\nu=j_{\nu-1}+1}^{n}\left\{\left[x-(a_{j_1}+a_{j_2}+\cdots+a_{j_\nu})\right]^+\right\}^n = \binom{n}{\nu}\cdot\left[(x-\nu a)^+\right]^n.$$

Eq. (12) follows from (9) by a similar argument.[1] □

Hereafter, for simplicity we shall take $a_n = a = 1$ for all $n \in \mathbb{N}$. Then, from Eq. (3) there follows

$$f_{n+1}(x) = F_n(x) - F_n(x-1), \quad \forall n \in \mathbb{N}, \forall x \in \mathbb{R} \tag{13}$$

so that

$$f_{n+1}(k) = F_n(k) - F_n(k-1), \quad \forall n \in \mathbb{N}, k \in \{0, 1, \ldots, n+1\}, \tag{14}$$

whereas from Eqs. (11) and (12) one obtains

$$F_n(k) = \frac{1}{n!}\sum_{\nu=0}^{k}(-1)^\nu \binom{n}{\nu}\left[(k-\nu)^+\right]^n, \quad \forall n \in \mathbb{N}, k \in \{0, 1, \ldots, n\} \tag{15}$$

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[1] Note that Eqs. (11) and (12) obtained by us as a special case of (8) and (9) are in agreement with a result due to Feller (1966).

and

$$f_n(k) = \frac{1}{(n-1)!} \sum_{\nu=0}^{k} (-1)^{\nu} \binom{n}{\nu} \left[(k-\nu)^+ \right]^{n-1}, \quad \forall n \in \mathbb{N}, k \in \{0, 1, \ldots, n\}. \tag{16}$$

Proposition 3.2. *When $a_n = a = 1$ for all $n \in \mathbb{N}$ then*

$$F_n(x) = \sum_{j=1}^{k} f_{n+1}(x + j - k), \quad \forall n \in \mathbb{N}, k \in \{1, 2, \ldots, n\}, k - 1 \leq x \leq k. \tag{17}$$

Proof. Starting from (13), by iteration it follows that

$$\begin{aligned} F_n(x) &= f_{n+1}(x) + F_n(x-1) = f_{n+1}(x) + f_{n+1}(x-1) + F_n(x-2) \\ &= \cdots = \sum_{j=1}^{k} f_{n+1}(x+j-k) + F_n(x-k). \end{aligned}$$

Since $x - k \leq 0$, one has $F_n(x-k) = 0$, which completes the proof. □

Proposition 3.3. *When $a_n = a = 1$ for all $n \in \mathbb{N}$ then*

$$\int_{k-1}^{k} F_n(x)\, \mathrm{d}x = F_{n+1}(k), \quad \forall n \in \mathbb{N}, k \in \{1, 2, \ldots, n\}. \tag{18}$$

Proof. Making use of (17) one obtains

$$\begin{aligned} \int_{k-1}^{k} F_n(x)\, \mathrm{d}x &= \sum_{j=1}^{k} \int_{k-1}^{k} f_{n+1}(x+j-k)\, \mathrm{d}x = \sum_{j=1}^{k} F_{n+1}(x+j-k) \Big|_{k-1}^{k} \\ &= \sum_{j=1}^{k} [F_{n+1}(j) - F_{n+1}(j-1)] = F_{n+1}(k) - F_{n+1}(0). \end{aligned}$$

The proof is then a consequence of $F_n(0) = 0$ for all $n \in \mathbb{N}$. □

Consider now the event $S_{n,k} = \{k - 1 \leq S_n \leq k\}$ and let $P_{n,k} := \mathbb{P}\left(S_{n,k}\right)$. From (14) it follows that

$$P_{n,k} = F_n(k) - F_n(k-1) = f_{n+1}(k), \quad \forall n \in \mathbb{N}, k \in \{1, 2, \ldots, n\}. \tag{19}$$

Lemma 3.1. *When $a_n = a = 1$ for all $n \in \mathbb{N}$ then*

$$\mathbb{P}\left(S_{n+1} \leq k, S_{n,k}\right) = F_{n+1}(k) - F_n(k-1), \quad \forall n \in \mathbb{N}, k \in \{1, 2, \ldots, n\}. \tag{20}$$

Proof. Let $n \in \mathbb{N}$ and $1 \leq k \leq n$. Then,

$$\mathbb{P}\left(S_{n+1} \leq k, S_{n,k}\right) = \mathbb{P}\left(X_{n+1} \leq k - S_n, S_{n,k}\right) = \iint_T f_{X_{n+1}}(x) f_n(y)\, \mathrm{d}x\, \mathrm{d}y$$

where T denotes the domain in the x–y plane defined by $0 < x < 1$ and $k - 1 < y < k - x$. Hence, by integration along the y-axis from $k-1$ to $k-x$, for all $x \in (0, 1)$ we obtain

$$\begin{aligned} \mathbb{P}\left(S_{n+1} \leq k, S_{n,k}\right) &= \int_0^1 \mathrm{d}x \int_{k-1}^{k-x} f_n(y)\, \mathrm{d}y = \int_0^1 F_n(k-x)\, \mathrm{d}x - F_n(k-1) \\ &= \int_{k-1}^{k} F_n(x)\, \mathrm{d}x - F_n(k-1). \end{aligned} \tag{21}$$

Eq. (20) follows from (21) and (18). □

Lemma 3.1 will be used to prove the following theorem.

Theorem 3.1. *When $a_n = a = 1$ for all $n \in \mathbb{N}$ then*

$$\mathbb{P}\left(S_{n+1} \leq k | S_{n,k}\right) = \frac{k}{n+1}, \quad \forall n \in \mathbb{N}, k \in \{1, 2, \ldots, n\}. \tag{22}$$

Proof. Let $k = 1$. $\forall n \in \mathbb{N}$, from (15) there follows $F_n(1) = 1/n!$, whereas $F_n(0) = 0$. Hence, making use of (19) one obtains

$$\mathbb{P}\left(S_{n+1} \leq 1 | S_{n,1}\right) = \frac{\mathbb{P}\left(S_{n+1} \leq 1, S_{n,1}\right)}{P_{n,1}} = \frac{F_{n+1}(1) - F_n(0)}{F_n(1) - F_n(0)} = \frac{n!}{(n+1)!} = \frac{1}{n+1}. \tag{23}$$

This proves (22) for $k = 1$. From (23) it follows that

$$\frac{1}{n+1} \equiv \mathbb{P}\left(S_{n+1} \leq 1 | S_{n,1}\right) = \mathbb{E}\left[\mathbb{P}\left(X_{n+1} \leq 1 - y | S_{n,1}, S_n = y\right)\right] = 1 - \mathbb{E}\left[S_n | S_{n,1}\right]$$

which ultimately implies

$$\mathbb{E}\left[S_n | S_{n,1}\right] = \frac{n}{n+1}.$$

Since $X_1, X_2, \ldots, X_n$ are uniform iid random variables, the mean of each of them conditional on $S_{n,1}$ is $1/(n+1)$. Hence, given that $S_{n,1}$ occurs, the means of $S_1, S_2, \ldots, S_n$ partition $[0, 1]$ into $n+1$ equally wide intervals. Therefore, for $1 < k \leq n$, if $S_{n,k}$ occurs, the interval that is partitioned into $n + 1$ equally wide intervals is now $[0, k]$. This implies that X_{n+1} cannot exceed $k/(n + 1)$ to insure that S_{n+1} remains below k. □

Corollary 3.1. *When $a_n = a = 1$ for all $n \in \mathbb{N}$ then*

$$\mathbb{E}\left[S_n | S_{n,k}\right] = \frac{n}{n+1} k, \quad \forall n \in \mathbb{N}, k \in \{1, 2, \ldots, n\}. \tag{24}$$

Proof. Due to Theorem 3.1 one has

$$\frac{k}{n+1} \equiv \mathbb{P}\left(S_{n+1} \leq k | S_{n,k}\right) = \mathbb{E}\left[\mathbb{P}\left(X_{n+1} \leq k - y | S_{n,k}, S_n = y\right)\right] = k - \mathbb{E}\left[S_n | S_{n,k}\right]$$

which ultimately yields Eq. (24). □

Corollary 3.2. *When $a_n = a = 1$ for all $n \in \mathbb{N}$ then*

$$\mathbb{E}\left[S_n 1_{S_{n,k}}\right] = \frac{n}{n+1} k P_{n,k}, \quad \forall n \in \mathbb{N}, k \in \{1, 2, \ldots, n\}. \tag{25}$$

Proof. Since

$$\mathbb{E}\left[S_n | S_{n,k}\right] = \frac{\mathbb{E}\left[S_n 1_{S_{n,k}}\right]}{P_{n,k}},$$

Eq. (25) follows from (24). □

Proposition 3.4. *When $a_n = a = 1$ for all $n \in \mathbb{N}$ then*

$$n \sum_{k=1}^{n} k f_{n+1}(k) = (n+1) \sum_{k=1}^{n} F_{n+1}(k), \quad \forall n \in \mathbb{N}, k \in \{1, 2, \ldots, n\}. \tag{26}$$

Proof. Let $n \in \mathbb{N}$. From (19) and (25) we obtain

$$\frac{n}{2} = \mathbb{E}\left[S_n\right] = \sum_{k=1}^{n} \mathbb{E}\left[S_n 1_{S_{n,k}}\right] = \frac{n}{n+1} \sum_{k=1}^{n} k f_{n+1}(k). \tag{27}$$

Making use of (18), we are easily led to

$$\mathbb{E}\left[S_n\right] \equiv \int_0^n x f_n(x)\, \mathrm{d}x = x F_n(x)\Big|_0^n - \int_0^n F_n(x)\, \mathrm{d}x = n - \sum_{k=1}^{n} \int_{k-1}^{k} F_n(x)\, \mathrm{d}x = n - \sum_{k=1}^{n} F_{n+1}(k).$$

Hence,

$$\frac{n}{2} = \sum_{k=1}^{n} F_{n+1}(k). \tag{28}$$

Eq. (26) finally follows by equating the right-hand sides of (27) and (28). □

The forthcoming recursive formulas are a consequence of Theorem 3.1.

Proposition 3.5. *When $a_n = a = 1$ for all $n \in \mathbb{N}$ then*

$$F_{n+1}(k) = F_n(k)\frac{k}{n+1} + F_n(k-1)\frac{n+1-k}{n+1}, \quad \forall n \in \mathbb{N}, k \in \{1, 2, \ldots, n+1\}. \tag{29}$$

Proof. Let $n \in \mathbb{N}$ and $1 \le k \le n+1$. From (19) and (20) one obtains

$$\mathbb{P}\left(S_{n+1} \ge k, S_{n,k}\right) = P_{n,k} - \mathbb{P}\left(S_{n+1} \le k, S_{n,k}\right) = F_n(k) - F_n(k-1) - F_{n+1}(k) + F_n(k-1),$$

or

$$\mathbb{P}\left(S_{n+1} \ge k, S_{n,k}\right) = F_n(k) - F_{n+1}(k). \tag{30}$$

On the other hand,

$$\mathbb{P}\left(S_{n+1} \ge k, S_{n,k}\right) = \mathbb{P}\left(S_{n+1} \ge k | S_{n,k}\right) P_{n,k} = \left[1 - \mathbb{P}\left(S_{n+1} \le k | S_{n,k}\right)\right] P_{n,k}.$$

Hence, from (19) and (22) one derives

$$\mathbb{P}\left(S_{n+1} \ge k, S_{n,k}\right) = F_n(k) - F_n(k-1) - \frac{k}{n+1}F_n(k) + \frac{k}{n+1}F_n(k-1). \tag{31}$$

Eq. (29) then immediately follows after equating the right-hand sides of (30) and (31). □

Note that (29) trivially holds also for $k = 0$, yielding $0 = 0$.

Remark 3.1. Since

$$\begin{aligned} \mathbb{E}\left[S_n 1_{S_{n,k}}\right] &= \int_{k-1}^{k} x f_n(x)\,\mathrm{d}x = x F_n(x)\Big|_{k-1}^{k} - \int_{k-1}^{k} F_n(x)\,\mathrm{d}x \\ &= kF_n(k) - (k-1)F_n(k-1) - F_{n+1}(k). \end{aligned}$$

Eq. (25) can be alternatively obtained via (29).

Proposition 3.6. *When $a_n = a = 1$ for all $n \in \mathbb{N}$ then*

$$P_{n+1,k} = P_{n,k}\frac{k}{n+1} + P_{n,k-1}\frac{n+2-k}{n+1}, \quad \forall n \in \mathbb{N}, k \in \{1, 2, \ldots, n+1\}. \tag{32}$$

Proof. Let $n \in \mathbb{N}$ and $1 \le k \le n+1$. By the difference of Eq. (29) written for k and for $k-1$, one obtains

$$P_{n+1,k} = P_{n,k}\frac{k}{n+1} + F_n(k-1)\frac{1}{n+1} + P_{n,k-1}\frac{n+1-k}{n+1} - F_n(k-2)\frac{1}{n+1},$$

whence (32) follows after noting that $F_n(k-1) - F_n(k-2) = P_{n,k-1}$. □

Proposition 3.7. *When $a_n = a = 1$ for all $n \in \mathbb{N}$ then*

$$f_{n+1}(k) = f_n(k)\frac{k}{n} + f_n(k-1)\frac{n+1-k}{n}, \quad \forall n \in \mathbb{N}, k \in \{1, 2, \ldots, n+1\}. \tag{33}$$

Proof. Let $n \in \mathbb{N}$ and $1 \le k \le n+1$. From (19) and (32) it follows that

$$f_{n+1}(k) = P_{n,k} = P_{n-1,k}\frac{k}{n} + P_{n-1,k-1}\frac{n+1-k}{n}.$$

By making again use of (19), Eq. (33) is finally obtained. □

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