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<td>$+x^1$</td>
<td>12/7</td>
<td>-6</td>
<td>2.127</td>
<td>22</td>
<td>9</td>
<td>21</td>
<td>53</td>
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<td>1.2414</td>
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<td>32</td>
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<td>27</td>
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<td>32</td>
<td>7</td>
<td>15</td>
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<tr>
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<td>50</td>
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<td>6.2413</td>
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<td>24</td>
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<td>15</td>
<td>11</td>
<td>59</td>
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<tr>
<td>$-x^9$</td>
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<td>2.156</td>
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<td>15</td>
<td>47</td>
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<td>23</td>
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<td>9</td>
<td>26</td>
<td>9</td>
<td>47</td>
</tr>
<tr>
<td>$-y^9$</td>
<td>25/9</td>
<td>-25/9</td>
<td>7.259</td>
<td>18</td>
<td>54</td>
<td>19</td>
<td>2</td>
<td>14</td>
<td>45</td>
<td>14</td>
<td>38</td>
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TRANSACTIONS
OF THE
CONNECTICUT ACADEMY
OF
ARTS AND SCIENCES.

VOLUME III.

NEW HAVEN:
PUBLISHED BY THE ACADEMY.
1874 to 1878.

Tuttle, Morehouse & Taylor, Printers, New Haven.
II. A List of Writings relating to the Method of Least Squares, with Historical and Critical Notes. By Mansfield Merriman.

The following list contains the titles of 408 papers, books and parts of books, relating to the Method of Least Squares and the Theory of accidental Errors of Observation, chronologically arranged according to their dates of publication. The first was issued in the year 1722 and the last in 1876. Previous to 1805, the year of Legendre’s announcement of the principle of Least Squares, there are 22 titles; since 1805 there is a continual yearly increase in the number of titles,—thus:

From 1805 to 1814 inclusive, there are 18 titles,

<table>
<thead>
<tr>
<th>Years</th>
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<tbody>
<tr>
<td>1805</td>
<td>18</td>
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<tr>
<td>1815</td>
<td>30</td>
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<td>1825</td>
<td>32</td>
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<td>1845</td>
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<td>1855</td>
<td>71</td>
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<tr>
<td>1865</td>
<td>95</td>
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These books and memoirs are in eight languages, and classified according to the place of publication, they fall under twelve countries. It may be interesting to note the number belonging to each, viz:

<table>
<thead>
<tr>
<th>Countries</th>
<th>Languages</th>
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<tbody>
<tr>
<td>Germany</td>
<td>German</td>
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<tr>
<td>France</td>
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<tr>
<td>Great Britain</td>
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<td>Latin</td>
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<td>Belgium</td>
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<td>Sweden</td>
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<tr>
<td>Denmark</td>
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<tr>
<td>Total</td>
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The 408 titles may be roughly classified as 313 memoirs, 72 books and 23 parts of books. They were written by 193 authors, 127 of whom produced only one book or paper each.
In preparing this list I have been able to use only the libraries of Yale College, (including that of the Connecticut Academy of Arts and Sciences,) but I think no work in those libraries relating to the subject has been left unconsulted. Out of the total 408 works, I have seen 312; the titles of these and the accompanying notes have been drawn from actual inspection. I have made no attempt to consult the literature of the Russian and Hungarian languages, and with wider library facilities the number of titles in the Italian, Dutch and Scandinavian languages would undoubtedly have been greater. Of course no work of this kind can be regarded as complete.

It has been my aim to record all writings which can be considered as contributions to the science of the Adjustment of Observations, and I think that those marked as actually inspected may be truly so regarded. Many works on Astronomy and Probability which devote but a page or two to the subject, as well as numerous practical papers, in which the Method of Least Squares is used incidentally and briefly, have been left unnoticed; to record all of these would be well nigh impossible, nor would the value of the list be thereby increased. Among the 96 which I have not seen there may possibly be a few that would be rejected after actual inspection.

The following is the arrangement of the list.

At the head of a title is placed the year of publication. In the case of memoirs this often differs from the date of the volume in which they are contained; for instance, Nautical Almanacs are published several years preceding and Transactions of Learned Societies often several years after the date which they bear. When a memoir is published in parts extending over two or more years it is recorded under the date of the first part.

The author's name follows the date of publication. In the index at the end of the list the full names of authors are given, and a distinction is made in the text when two persons of the same surname have written on the subject.

The titles of books, and pamphlets published as books, are printed in *italics*, and the titles of memoirs in ordinary type. Those which I have actually inspected have their titles enclosed in single quotation marks ('---') and these are intended to correspond with the originals in punctuation, spelling and when possible in the use of capitals.

The place of publication of books is given, with references to subsequent editions or translations. The usual terms 4to, 8vo, etc. are added, although they give little idea of the size of a book, and the
number of pages is noted whenever I have been able to ascertain it. When a single chapter only of a book relates to the subject, the title of that chapter is treated like a memoir.

After the title of a memoir is placed in italics the title of the volume containing it. This is abridged in the usual manner, but the words of the title are never transposed. For instance, Bulletin des Sciences de la Société Philomathique de Paris is abridged into Bull. Soc. Philom. Paris. In a few instances I have added the place of publication or have prefixed the name of the editor in order to ensure perfect clearness. The number of the volume and the pages which are devoted to the memoir are given; the mention of the year renders it unnecessary to note the various series.

When the work was begun it was intended to make the notes very full so as to give a tolerably complete history of the Method of Least Squares. But as the number of titles began to multiply under research it became evident that the plan would produce a manuscript too voluminous for publication. The notes were hence abbreviated into their present form. The work begun as historical has, I am afraid, ended by being largely bibliographical.

Sometimes the notes give an account of the contents of a memoir or an estimate of its value; sometimes they take the form of a direct quotation from the memoir itself or state the opinion of some subsequent reviewer; and occasionally they offer critical remarks of my own. But always they aim to give such cross references as will enable the student to follow up special lines of investigation and gain the fullest information concerning a particular memoir or book. Brief as the notes are, I hope they will be found at least suggestive by those who use them. To the future historians of mathematical science they will undoubtedly be of very great value.

The mode of cross reference usually adopted is to mention simply the year and author. Thus “1818 Bessel” refers either to a book published in 1818 by Bessel or to notes under that heading.

The following table points out some of the most valuable papers on the proofs of the Method of Least Squares:

First publication of the Method.......................... see 1805 LEGENDRE.
First and Second Proofs............................... 1808 ADRAIN.
Third Proof........................................... 1809 GAUSS.
Fourth Proof........................................... 1810 LAPLACE.
Fifth Proof............................................ 1812 LAPLACE.
Theory and Practice compared....................... 1818 BESSEL.
Sixth Proof............................................ 1823 GAUSS.

Seventh Proof. ........................................ 800 1825 IVORY.
Eighth Proof ........................................ 1826 IVORY.
Proof of the Arithmetical Mean ..................... 1832 ENCKE.
Ninth Proof .......................................... 1837 HAGEN.
Tenth Proof .......................................... 1838 BESSEL.
LAPLACE's Proof extended and improved ............. 1844 ELLIS.
Eleventh Proof ...................................... 1844 DONKIN.
ADRAIN's Second Proof rediscovered ................. 1850 HERSCHEL.
Twelfth Proof .................................... 1856 DONKIN.
On the Arithmetical Mean ............................ 1864 DEMORGAN.
[Thirteenth Proof] ................................ 1870 CROFTON.
Analysis of several Proofs .......................... 1872 GLAISHER.
Proof of the Arithmetical Mean ...................... 1875 SCELIPARELLI.

I have drawn information from every source within my reach. On
the Proofs of the Method I have found GLAISHER's memoir of 1872
of the greatest value, and while working on the early writers TOD-
HUNTER's History of Probability was continually before me. It is
here also the place to acknowledge my indebtedness to Prof. H. A.
NEWTON of Yale College for valuable suggestions and kind assistance.

1722 COTES. 'Æstimatio errorum in mixta mathesi, per variationes
partium trianguli plani et sphærici.' Opera miscellanea (appended
to Harmonia mensurarum; Cantabrigiae, 4to), pp. 1–22. —Memoir
repubhshed, Lengoviae, 1768, 8vo.

Only the closing paragraph relates to accidental errors of observa-
tion; this gives the following rule: "Sit p locus Objecti alicujus ex
Observatione prima definitus, q, r, s ejusdem Objecti loca ex Obser-
vationibus subsequentibus; sint insuper P, Q, R, S pondera reciproce
proportionalia spatii Evaginationum, per quae se diffundere possint
Errores ex Observationibus singulis prodeuentes, quasee dantur ex
datis Errorum Limitibus; & ad puncta p, q, r, s posita intelligentur
pondera P, Q, R, S, & inventariar eorum gravitatis centrum Z: dico
punctum Z fore Locum Objecti maxime probabilimum, qui pro vero ejus
loco tutissime haberi potest."

COTES's rule only agrees with modern methods when the observa-
tions are directly made upon one quantity. See LAPLACE, Théorie
analytique des Probabilités, third edition, p. cxxxviii, p. 346; and
IVORY, Phil. Mag., 1825, Vol. LXV, p. 4.

1749 EULER. Pièce qui a remporté le prix de l'Academie royale
des sciences en 1749, sur les inegalités du mouvement de Saturne et de
Jupiter. Paris, 4to.

Contains a method for the combination of linear equations similar
to the following.

By twenty-seven observations upon the position of a moon spot Mayer obtained twenty-seven equations each containing three unknown quantities. To solve these he added together those nine equations in which the values of the coefficients of one of the unknown quantities were the greatest, then the nine in which these coefficients were the least and lastly the remaining nine; thus obtaining three resulting equations with three unknown quantities.


Treats of errors in Surveying and probably contains nothing of value on the theory of accidental errors. See also the same memoirs for 1768, p. 147 and p. 159.

1755 Boscovich. *De littera expeditione per Pontificam ditionem ad dinetiendos duos meridiani gradus.* Romae, 4to, pp. xxii, 516. —French translation by Hugon entitled *Voyage astronomique et geographique...*; Paris, 1770, 4to.


1756 Simpson. 'A letter to the Right Honourable George Earl of Macclesfield, President of the Royal Society, on the Advantage of taking the Mean of a number of Observations, in practical Astronomy.' *Phil. Trans. Lond.* for 1755, Vol. XLIX, Pt. I, pp. 82–93. —Reprint, see 1757.

This memoir is interesting and valuable as being the first in which the Theory of Probability is applied to the discussion of errors of observation and in which the idea of a law of facility of error is implied. At the beginning of the letter Simpson says that his attention had been called to the subject by the fact that "some persons, of considerable note, have been of opinion, and even publicly maintained, that one single observation, taken with due care, was as much to be relied on as the Mean of a great number."

The letter contains two propositions; the first gives a method of determining the probability that the error of the mean of *n* observations shall be less than an assigned value, provided it is equally prob-
able that the error of a single observation may be any one of the quantities, \(-v, \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots, v\). The second gives a method of determining that probability, provided the probabilities of the single errors \(-v, -v+1, \ldots, -1, 0, 1, \ldots, v-1, v\) are proportional to the terms 1, 2, \ldots, \(v, v+1, \ldots, 2, 1\). This is illustrated by a numerical example. Simpson remarks that the advantage of the mean can be shown, whatever series be used to express the chances of the errors.

1757 Simpson. ‘An Attempt to show the Advantage arising by Taking the Mean of a Number of Observations, in practical Astronomy.’ *Miscellaneous Tracts* . . . (London, 4to), pp. 64–75.

A reprint of the preceding, the opening and closing paragraphs being omitted and nearly four pages of new matter added. We find here the following axioms stated for the first time; 1. that positive and negative errors are equally probable, 2. that there are certain assignable limits within which all errors may be supposed to fall.

In the added matter the case of *continuous* errors is discussed, and the probability that the error of the mean is less than an assigned value found for the case of the second proposition by making \(v\) and other quantities in the formulæ infinite. Simpson represents the law of facility of error geometrically by the sides of an isosceles triangle and draws a curve to show the increased precision of the mean as compared with single observations. He closes by finding under the same supposition as to the law of facility, the probability that the mean is nearer to the truth than a single observation taken at random. The whole memoir must have been extremely valuable at the time of its publication.


A method of combining discordant observations upon the lengths of degrees of the earth’s meridian is here given. The adjustment is effected under the two conditions that the sum of the negative errors shall be equal to the sum of the positive errors, and that each sum shall have the least possible value. The problem was solved by a geometric construction depending upon the properties of the centre of gravity of figures.


1760 Lambert. *Photometria sive de mensura et gradibus luminis* . . . Augustae Vindelicorum, 8vo, pp. [xxx], 547.

Contains many remarks on the arithmetical mean and also proposes a method for judging of the precision of the measurements, which
consists in comparing the mean with a new mean found after rejecting that observation deviating most from the first mean. See *Nova Acta Lipsiae*, 1760, p. 560; also Lambert’s *Beyträge...* Vol. I, p. 426.


Contains a method of adjusting simple observations founded on the principle that the algebraic sum of the errors shall be zero. The method is illustrated by the determination of empirical formulae for the length of the seconds pendulum, the declination of the magnetic needle, etc.


This is Part V of the ‘Mémoire sur la probabilité des causes par les évènements,’ which occupies pages 621–656 of the volume. It contains the first attempt to deduce a rule for the combination of observations from the principles of Probability.

Laplace begins by saying that the law of probability of errors of observation may be represented by a curve whose equation is \( y = \psi(x) \), \( x \) being any error and \( y \) its probability; and this curve must have three properties: 1st, it must be symmetrical with reference to the axis of \( y \), since positive and negative errors are equally probable; 2nd, the axis of \( x \) must be an asymptote, since the probability of the error \( \infty \) is 0; 3rd, the area of the curve must be unity, since it is certain that an error will be committed.

Laplace takes \( \psi(x) \) as \( \psi(x) = \frac{1}{2} me^{-mx} \) (\( x \) being regarded as always positive), but his reasons for doing so are slight. With this law he finds the mean of three observations, regarding it as corresponding to an ordinate which divides a curve \( u = \psi(x_1)\psi(x_2)\psi(x_3) \) into two equal parts. His result is as follows: Let \( M_1, M_2 \) and \( M_3 \) be the three measurements, of which \( M_1 \) is the least; let \( M + x \) be the mean; it is required to find \( x \). Put \( M_2 - M_1 = p \) and \( M_3 - M_2 = q \); then \( x \) is given by

\[
x = p + \frac{1}{m} \log \left( 1 + \frac{1}{3} e^{-mr} - \frac{1}{3} e^{-ms} \right),
\]

in which \( m \) is a constant depending upon the precision of the observation. Laplace then shows that this value cannot agree with the rule of the arithmetical mean, and he computes a table for finding \( x \) for certain given ratios of \( q \) to \( p \). For instance

<table>
<thead>
<tr>
<th>( q )</th>
<th>( p )</th>
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<tbody>
<tr>
<td>0.0</td>
<td>0.860</td>
</tr>
<tr>
<td>0.1</td>
<td>0.894</td>
</tr>
<tr>
<td>0.2</td>
<td>0.916</td>
</tr>
<tr>
<td>0.3</td>
<td>0.932</td>
</tr>
<tr>
<td></td>
<td></td>
</tr>
</tbody>
</table>
Thus if three measurements of an angle give
\[ M_1 = a^0 b' 0'', \quad M_2 = a^0 b' 40'' \quad \text{and} \quad M_3 = a^0 b' 50'', \]
we have \( p=40'', q=10'' \) and \( q=0.25p \); then from the table \( x=37'' \) and the adjusted result is \( a^0 b'37'' \), while by the usual rule of the average we would have \( a^0 b'30'' \). By Laplace's table the mean will lie nearer to the two observations which most nearly agree, than in the common method.


[1774] Lagrange. 'Mémoire sur l'utilité de la méthode de prendre le milieu les résultats de plusieurs observations; dans lequel on examine les avantages de cette méthode par le calcul des probabilités; & où l'on résoud différens problèmes relatifs à cette matière.' Miscellanea Taurinensia (Mé. Soc. Turin) for 1770-1773, Vol. V, pp. 167-232 of the math. part.

This memoir is a more thorough presentation of the subject treated by Simpson in 1757 with much new matter added. Lagrange makes no allusion however to previous writings on the subject. The expression "Law of Facility of Error" occurs here for the first time.

For an extended account of the contents of the memoir see Todhunter, History of Probability, pp. 301-313. See also below 1785 Bernoulli, 1788 Euler, and 1804 Tremain. In 1850 Encke gives a translation of part of the memoir, with comments.


The now familiar illustration of a marksmen firing at a target is here introduced, and the conclusion drawn that small errors are more probable than large ones, and that the method of taking the arithmetical mean "non sine ratione dubitare potest," since it supposes the observations of equal weight.

Daniel Bernoulli takes a circle \( y = \sqrt{r^2-x^2} \) as representing the law of facility of error, \( y \) being proportional to the probability of the error \( x \); and \( r \) a constant. Then if observations give the errors \( x_1, x_2, x_3, \ldots \), the product \( \sqrt{r^2-x_1^2} \sqrt{r^2-x_2^2} \sqrt{r^2-x_3^2} \ldots \) must be a maximum to give the most probable value of the observed quantity. He finds that this value coincides with that given by the rule of the arithmetical mean for one and for two observations, and that it nearly coincides for three when a suitable value is given to \( r \). For a greater number than three his method leads to unmanageable equations. He closes by remarking that the problem is indeterminate. See Zach's Monatliche Correspondenz, 1805, Vol. XI, pp. 486-490.
Daniel Bernoulli's method agrees closely with modern theory. See 1778 Euler, 1785 Bernoulli and Todhunter, History of Probability, p. 236.


Euler considers that Daniel Bernoulli is correct in objecting to the arithmetical mean when the observations are of unequal precision, but that he "was quite arbitrary in proposing to make the product of the probabilities a maximum. Euler proposes another method which amounts to making the sum of the fourth powers of the probabilities a maximum."—Todhunter, Hist. of Probability, p. 237.


Pages 322–332 are devoted to the discussion of the mean to be taken between discordant observations. Laplace says that by the expression mean or mean result an infinite number of things may be understood. Of these the one which is implied in adjusting observations is a value such that the resulting error shall be a minimum, and this corresponds to a value such that the sum of the errors, each multiplied by its probability shall be a minimum. The method of his memoir of 1774 is then substantially repeated.

Laplace remarks that if ±a be the limits of error, the law of facility ought to be \( \varphi(x) = \frac{1}{2a} \log_e \frac{a}{x} \). His methods lead to unmanageable equations.


Boscovich's method of 1760 and Lambert's of 1765 are referred to, and an account is given of Daniel Bernoulli's memoir of 1778, which differs slightly from the memoir itself. An account of Lagrange's memoir of 1774 is presented with considerable fullness. See Todhunter, Hist. of Probability, p. 442.

1788 Euler. 'Eclaircissements sur le mémoire de Mr De La Grange inséré dans le V° volume de Mélanges de Turin, concernant la méthode de prendre le milieu entre les résultats de plusieurs observations, &c.' Nova Acta Acad. Petrop. for 1785, Vol. III, pp. 289–297 of the memoirs.

The memoir seems to have no value; See Todhunter, Hist. of Probability, p. 250.

The method of *indeterminate corrections*, afterwards of so much use in the computations of Least Squares is here stated and used, although perhaps not for the first time. See 1820 Legendre.


The matter of this memoir is mostly reproduced in Sections 39–42 of Chap. V, Book III, of the *Traité de mécanique céleste*.

First a method is presented of determining an elliptic meridian so that the greatest error shall be a minimum; see 1831 Cauchy. Secondly a method is developed for finding an ellipse subject to the following conditions: “1° que la somme des erreurs soit nulle; 2° que la somme des erreurs prises toutes avec le signe + soit un minimum.” Laplace mentions that Boscovich (1760) had solved the same problem by a different method. He alludes to the ellipse thus determined as “l’ellipse la plus probable.”

1799 Kramp. ‘Table première. Intégrales de $e^{-dt}$, depuis une valeur quelconque de $t$ jusqu’à $t$ infinie.’ and ‘Table seconde. Logarithmes des intégrales $fe^{-dt}$.’ *Analyse des refractions...* (Strasbourg, 4to), pp. 195–206.


“Appears to be of no value whatever.”—Todhunter, *Hist. of Probability*, p. 428.


Contains applications of Laplace’s methods of 1792 to the deduction of empirical formulæ from discordant observations.

1805 Legendre. ‘*Nouvelles méthodes pour la détermination des orbites des comètes.*’ Paris, 4to, pp. viii, 80. —Second edition, see 1806.

The date 1806 is generally given for this book, in which year a second edition seems to have been issued. The copy before me is plainly dated, “An XIII—1805.”
In the preface Legendre gives an outline of his method for computing orbits. On page viii he says: "Il faut ensuite, lorsque toutes les conditions du problème sont exprimées convenablement, déterminer les coefficients de manière à rendre les erreurs les plus petites qu'il est possible. Pour cet effet, la méthode qui me paraît la plus simple et la plus générale, consiste à rendre minimum la somme des carrés des erreurs. On obtient ainsi autant d'équations qu'il y a de coefficients inconnus; ce qui achève déterminer tous les éléments de l'orbite.....la méthode dont je viens de parler, et que j'appelle Méthode des moindres quarrés, peut être d'une grande utilité....."

On page 84 is an application of the method to the solution of three equations with two unknown quantities, in which the now well-known rule for the solution of normal equations is followed. On page 68 and 69 are references to its use. Pages 72-70 constitute an Appendix "Sur la Méthode des moindres quarrés." In this, after having mentioned that it is impossible that the sum of the errors should be zero when the number of given equations exceeds that of the unknown quantities, Legendre says: "De tous les principes .... je pense qu'il n'en est pas de plus général, de plus exact, ni d'une application plus facile que celui .... qui consiste à rendre minimum la somme des quarrés des erreurs. Par ce moyen, il s'établit entre les erreurs une sorte d'équilibre qui empêchant les extrêmes de prévaloir, est tres-propre à faire connaître l'état du système le plus proche de la vérité."

Legendre then proceeds to deduce the rule for the formation of "l'équation du minimum par rapport à l'une des inconnues," or as we now say of a normal equation. His notation is the following: in the \( n \) equations

\[
\begin{align*}
0 &= a + bx + cy + \ldots \\
0 &= a' + b'x + c'y + \ldots \\
0 &= a'' + b''x + c''y + \ldots \\
\ldots & \ldots \ldots \ldots \ldots
\end{align*}
\]

\( a, b, c, \ldots, a', b', c' \ldots \) are known by observation or theory, and \( x, y, \ldots \) are to be determined. By forming the sum of the squares of these equations, differentiating with reference to each unknown separately and placing the derivatives equal to zero, he finds

\[
\begin{align*}
0 &= \int ab + x'b^2 + y'bc + \ldots \\
0 &= \int ac + x'b'c + y'c^2 + \ldots \\
\ldots & \ldots \ldots \ldots \ldots
\end{align*}
\]

which are the same in number as the unknown quantities \( x, y, \ldots \), and in which

\[
\begin{align*}
\int ab &= ab + a'b' + a''b'' + \ldots \\
\int b^2 &= b^2 + b'^2 + b''^2 + \ldots \\
\ldots & \ldots \ldots \ldots \ldots
\end{align*}
\]

Legendre next demonstrates that the rule of the arithmetical mean is a particular case of his general principle. He then supposes the position of a point in space to be determined by three observations and finds the values of its coördinates given by the method. Noticing their identity with those for the centre of gravity of three points in space, he announces that the sum of the squares of the distances of
all the molecules of a body from its centre of gravity must be a minimum, and that hence "la méthode des moindres quarrés fait connaître, en quelque sorte, le centre autour duquel viennent se ranger tous les résultats fournis par l'expérience, de manière à s'en écarter le moins qu'il est possible." A numerical example of the application of the method to the determination from five observations of the form of a meridian of the earth closes the book.

The honor of the first publication of the Method of Least Squares belongs to Legendre. Although he failed to claim that his method gives most probable or most advantageous results, yet the remarks above quoted indicate that he fully recognized it as a rule giving a plausible and reasonable mean. See below 1814.


1806 Legendre. 'Supplément aux Nouvelles Méthodes pour la détermination des orbites des comètes,' Paris, 4to, pp. 55.

The copy of this supplement which is before me is preceded by a title page reading, "Nouvelles méthodes pour la détermination des orbites des comètes; avec un Supplément.... A Paris, ... Année 1806." This title page was probably prefixed to copies of the work of 1805, with the Supplement added, as the date 1806 is usually stated for it. A second Supplement appeared in [1820].

On pages 28 and 43 the Method of Least Squares is used.


Laplace's method of 1792 is stated and used in the discussion of the most probable elliptic meridian. The method is attributed to Boscovich; see 1760.

On pages 138–143 "die....von Legendre vorgeschlagenen Methode, die er Méthode des moindres quarrés nennt," is explained and applied in the determination of the elliptic meridian. The date of Legendre's *Nouvelles méthodes....* is given as 1806.


On page 184 Gauss says: "Legendre's Werk....habe ich noch nicht gesehen. Ich hatte mit Fleiss mir deswegen keine Mühe gegeben, um bey der Arbeit an meiner Methode ganz in der Kette meiner eigenen Ideen zu bleiben. Durch ein paar Worte, die de la Lande in der letztern Historie de l'Astronomie, 1805, fallen lässt,
méthode des moindres quarrés, gerathe ich auf die Vermuthung, dass ein Grundsatz, dessen ich mich schon seit zwölf Jahren bey maucherley Rechnungen bedient habe, und den ich auch in meinem Werke mit gebrauchen werde, ob er wol zu meiner Methode eben nicht wesentlich gehört,—dass dieser Grundsatz auch von Legendre benutzt ist."

The work of Legendre here alluded to is the Nouvelles méthodes ....1805, and that of Gauss the Theoria motus ...1809, then in preparation. Gauss mentions some of the advantages of his method for computing orbits but gives no hint of the principle of Least Squares.

1808 Bowditch. ‘Solution of Mr. Patterson’s Prize Question for correcting a survey, proposed in No. II. page 42, No. III. page 68, by Nathaniel Bowditch, to whom the Editor has awarded the prize of ten dollars.’ The Analyst or Math. Museum, Vol. I, pp. 88–92.

The Prize Question was: "In order to find the content of a piece of ground, ....I measured, with a common circumerenter and chain, the bearings and lengths of its several sides, .... But upon casting up the difference of latitude and departure, I discovered ....that some error had been contracted in taking the dimensions. Now it is required to compute the area of this enclosure, on the most probable supposition of this error."

Bowditch's solution depends on several "principles" or hypotheses, the chief of which is "that in measuring the lengths of any lines the errors would probably be in proportion to their lengths." No principles of the Theory of Probability are employed except such as are by common sense implied. His solution coincides with that given by the Method of Least Squares.

This Prize Question undoubtedly led to the following 'Research' by Adrain, the Editor of The Analyst.


This paper seems to have been unknown to mathematicians until 1871 when it was partly reprinted in Amer. Jour. Sci., see 1871 ABBE. It is of great historical interest as containing the first deduction of the law of facility of error

\[ q(x) = ce^{-h^2x^2} \]

\( q(x) \) being the probability of any error \( x \), and \( c \) and \( h \) constants depending upon the precision of the measurement. The term "Least Squares" is not used, and Adrain seems to have been entirely unacquainted with Legendre’s writings.

Adrain gives two deductions of this law. The first, occupying pages 93–15 has been reprinted as noted above and need not here be repeated. It depends upon the "self-evident principle" that the true errors of measured quantities are proportional to the quantities them-
selves. The arbitrary nature of this assumption is shown by Glaisher, Mem. Astron. Soc. Lond., 1872, Vol. XXXIX, pp. 75–81, where the proof is analyzed and regarded as "very slight and inconclusive."

The second proof occupies pages 96–97, and not having been alluded to in the reprint noted above, I give it in A Drain's own words as The Analyst is quite rare:

"Suppose that the length and bearing of $AB$ are to be measured; and that the little equal straight lines $Bb$, $Be$ are the equal probable errors, the one $Bb=Bb'$ of the length of $AB$, and the other $Be=Be'$ (perpendicular to the former) of the angle at $A$, when measured on a circular arc to the radius $AB$; and let the question be to find such a curve passing through the four points $b$, $c$, $b'$, $c'$, which are equally distant from $B$, that, supposing the measurement to commence at $A$, the probability of terminating on any point of the curve may be the same as the probability of terminating on any one of the four points $b$, $c$, $b'$, $c'$.”

Then follows trivial reasoning which ends by concluding that "the curve must be the simplest possible" and "must consequently be the circumference of a circle having its centre in $B." This established, the proof is the following:

"Now let us investigate the probability of the error $Bm=x$, and of $nm=y$. Let $X$ and $Y$ be two similar functions of $x$ and $y$ denoting those probabilities, $X'$, $Y'$, their logarithms, then $XX'Y$ constant, or $X' + Y' = constant$, and therefore $X' + Y' = 0$, or $X'x + Y'y = 0$, whence $X'x = -X'y$. But $x^2 + y^2 = r^2 = Bb^2$, therefore $xx = -yy$, by which dividing $X'x = -Y'y$, we have $X'x = y'$; and therefore, by a fundamental principle of similar functions, the similar functions $X'$ and $Y''$ must be each a constant quantity: put then $X'x = n$, and we have $X'x = nxx$, that is $X' = nx$, and the fluent is $X' = c + \frac{nx^2}{2}$; in like manner we find $Y' = c + \frac{ny^2}{2}$ and therefore the probabilities themselves are $e^{c+\frac{nx^2}{2}}$ and $e^{c+\frac{ny^2}{2}}$, in which $n$ ought to be negative, for the probability of $x$ grows less as $x$ grows greater."

I have seen no allusion to this proof by any subsequent writer. It is essentially the same as given in 1850 by Herschel and usually called Herschel's proof. I regard it as defective in taking "$XX'Y$ constant," or in considering the probabilities of the $x$ and $y$ deviations as independent. See 1850 Ellis. See Boole's Finite Differences (Cambridge, 1860), pp. 228–229.
In his first proof Adrain had found for the probability of the error $x$ in the observed value $a$ the expression $e^{-\frac{2ax}{a+b+c}}$, and had shown that the most probable values of the observed quantities $a, b, c, \ldots$ must satisfy the condition

$$\frac{x^2}{a} + \frac{y^2}{b} + \frac{z^2}{c} + \ldots = a \text{ minimum.}$$

This principle he applies to four problems; the first showing that the arithmetical mean is a particular case of the method, the second to determine the most probable position of an observed point in space, which is shown to be "precisely in the centre of gravity of all the given points;" the third "to correct the dead reckoning at sea," and the fourth "to correct a survey."

As remarked by Abbe "we must credit Dr. Adrain with the independent invention and application of the most valuable arithmetical process that has been invoked to aid the progress of the exact sciences."—*Amer. Jour. Sci.*, 1871, Vol. I, p. 415.


That demonstration of the Method of Least Squares usually called Gauss’s proof or Gauss’s first proof is here presented. Assuming that the arithmetical mean of direct observations is the most probable value of the measured quantity, it deduces that the law of facility of error is given by

$$\varphi(x) = ce^{-\frac{1}{2}x^2}$$

from which the principle of Least Squares at once follows. This proof has been adopted by the majority of books on the subject; see for instance 1832 Encke, 1857 Dienger, 1858 Ritter, 1864 Chauvenet, 1867 Hansen and Merriman’s *Elements of the Method of Least Squares* (London, 1877, 8vo).

The demonstration as given by Gauß contains three defects. 1. It is not recognized that the probability of a definite error $x$, is an infinitesimal; this is avoided by some later writers. 2. The distinction between true errors and residuals (or calculated errors) is not sharply drawn; according to Gauß’s reasoning the law $\varphi(x) = ce^{-\frac{1}{2}x^2}$ is not strictly a "law of facility of error" but only a law of distribution of residuals. 3. The rule of the arithmetical mean is assumed. For critical analyses of this proof see below, 1843 Reuschle, 1844 Ellis and 1872 Glaisher.

Practical features of the method,—the formation of normal equations and the determination of weights and degrees of precision are also discussed in the Section and hints are given regarding its use in astronomy. On page 221 is an attempt to justify the principle of
Least Squares on the ground that any other process would lead to impracticable calculations.

I quote a historical remark on page 221: “Ceterum principium nostrum, quo jam inde ab anno 1795 usi sumus, nuper etiam a clar. LEGENDRE in operé Nouvelles méthodes ...comètes, 1806 prolatum est.....”

1810 BESSEL. Untersuchungen über die scheinbare und wahre Bahn der grossen Cometen von 1807. Königsberg, 4to.

The Method of Least Squares is used in determining the orbit and is called “den moindres quarrés.” See Zach’s Monatl. Corres., 1810, Vol. XXII, pp. 205-212.


Pages 383-389 and 559-565 are devoted to the Theory of Errors, and the principle of Least Squares is proved “lorsque les résultats entre lesquels on doit prendre un milieu, sont donnés chacun par un très-grand nombre d’observations, quelles que soient d’ailleurs les lois de facilité des erreurs de ces observations.”


Twelve equations involving six unknown corrections to the elements of the orbit are solved by the Method of Least Squares. We here find for the first time the notation

\[ [ab] = a' b' + a'' b'' + a''' b''' + \ldots \]

and also the algorithm for the solution of normal equations by successive substitution, since universally followed in lengthy computations.


Contains “the theory of errors substantially coincident with so much of the same theory as we find in pages 314-328 and 340-342 of the Théorie ... des Prob.” — TODHUNTER, History of Probability, p. 490. See below 1812.


Contains matter which is reproduced in the *Théorie... des Prob.*, 1812, pp. 322–329.


"... the greater part of the *Théorie des Probabilités* is a reprint of papers in the Memoirs of the Academy, which appear to contain the contents of the first papers on which he set down his processes. These with preliminary chapters, descriptive not of what follows, but of the general methods which he drew from the following parts, make up the whole work."—De Morgan, *Theory of Probabilities in Encyc.* Mérop., p. 453. It "is by very much the most difficult mathematical work we have met with."—*Ibid.*, p. 418. Todhunter in his *History of Probability*, p. 560, Ellis (1844) and other writers have also testified to the abstruseness of Laplace's methods.

The Method of Least Squares is developed in Chap. IV, (pages 304–348) of the second part of the work. The analysis only extends to the case of two unknown quantities or elements, and the number of observations is required to be very large or infinite. Under these restrictions the Method is shown to give most advantageous results, whatever be the law of facility of error provided only that positive and negative errors are equally probable. Laplace's definition of *most advantageous results* is the following: "...si l'on multiplie les erreurs possibles d'un élément par leurs probabilités respectives, le système le plus avantageux sera celui dans lequel la somme de ces produits tous pris positivement, est un minimum." The results thus obtained are not necessarily the *most probable*.

In the concluding paragraph of Chap. IV and in the opening pages of the First Supplement (see 1815), Laplace has given a general account of his method of analysis. These remarks and the table of contents at the end of the volume, give a much clearer idea of the steps of the demonstration than does Chap. IV itself. The principal objection against the validity of the proof is that it requires an infinite or very large number of observations. With this requirement, however, Gauss’s proof of 1809 becomes perfectly logical and the results are the *most probable*, not merely *most advantageous*. 
LAPLACE's proof has been greatly improved by subsequent writers. Ellis in 1844 extended it to any number of unknown quantities, Todhunter in his History of Probability, pp. 560–588 supplied a valuable commentary, and Glaisher in 1872 presented it in a clear and simple form. See also below, 1824 Poisson, 1847 DEMORGAN, 1852 BIENAYMÉ, 1861 AIRY, 1873 LAURENT and 1875 Diengerr.

On pages 318–319 is given what is sometimes called LAPLACE's second proof of the Method of Least Squares. This depends on the definition that "la valeur moyenne de l'erreur a croître en plus" should be a minimum. The reasoning is similar to GAUSS's proof of 1823; see Ellis's and Glaisher's papers quoted below under 1844 and 1872.


1814 CAUCHY. Mémoire sur le système de valeurs qu'il faut attribuer à divers Éléments, determineés par un grand nombre d'observations. [Paris, Lith. MS.]

Probably the same as his memoir of 1831.


Near the end are some remarks concerning the history of the Theory of Errors of observation, and descriptive of LAPLACE's processes.

1814 LEGENDRE. 'Méthode des moindres quarrés, pour trouver le milieu le plus probable entre les résultats de differentes observations.' Mém. Inst. France for 1810, Pt. II, pp. 149–154.

Pages 72–75 of the Nouvelles méthodes....1805, are here quoted and reference made to the practical applications of the method given in that work, in order to call attention to LEGENDRE's priority of publication.


We here find the first mention of probable error. After giving 48 observations on the right ascension of Polaris whose arithmetical mean is 55° 48'.5104, Bessel says: "Der wahrscheinliche Fehler einer einzelnen Beobachtung ist, nach den wirklichen vorkommenden Fehlern zu urtheilen =1'.067, und daher der wahrscheinliche Fehler des Endresultats =0'.154. Die Grund dieser Schätzung des wahrscheinlichen Fehlers, beruhen auf der von GAUSS gegebenen Entwickelung der Wahrscheinlichkeit, einen Fehler von gegebener
Größre zu begehen; ihre Mittheilung muss ich bis auf eine andere Gelegenheit versparen.” See 1816 Bessel, and also page 196 of the memoir 1816 Gauss.


This is devoted partly to a general description of Laplace’s proof of the Method of Least Squares, and partly to the discussion of the probability of results obtained by that method; a numerical example illustrates the use of his formula. See Todhunter, Hist. of Probability, p. 610; also see 1869 Todhunter.


Treats of Laplace’s method of adjustment of 1792.


Bessel defines the probable error as follows: “Ich verstehe unter dieser Benennung die Grenze, die eine Anzahl kleinerer Fehler von einer gleichen Anzahl grösserer trennt, so dass es wahrscheinlicher ist, eine Beobachtung innerhalb jeder weiteren Grenze von der Wahr- heit abirren zu sehen, als ausserhalb derselben.” If we designate by $\frac{\Sigma x}{n}$ the mean of the errors all taken positively, by $\frac{\Sigma x^2}{n}$ the mean of the squares of those errors, and by $r$ the probable error of a single observation, his demonstration shows that

$$r = 0.8453 \frac{\Sigma x}{n} \text{ or } r = 0.6745 \sqrt{\frac{\Sigma x^2}{n}}.$$  

Bessel does not distinguish between true errors and residuals. These formulae he uses in finding the probable errors of the elements of the orbits, which are deduced by the help of the Method of Least Squares.


This memoir gives three methods for finding the probable error from given observations. The first, which is that usually presented in text-books, finds that $r$ the probable error of an observation of the weight unity, is given (most probably) by

\[ r = 0.6744897 \sqrt{\frac{\sum x^2}{n}}, \]

\( \Sigma x^2 \) being the sum of the squares of the errors and \( n \) the number of observations, and that it is an even wager that the true value of \( r \) lies between

\[
\frac{0.6744897 \sqrt{\frac{\sum x^2}{n}}}{1 - \frac{0.4769363}{\sqrt{n}}} \quad \text{and} \quad \frac{0.6744897 \sqrt{\frac{\sum x^2}{n}}}{1 + \frac{0.4769363}{\sqrt{n}}}
\]

In the second method the most probable value of the sum \( \Sigma x^2 \) is discussed and formulae for probable error found when \( m \) has the values 1, 2, 3, 4, 5 and 6. The second of these, which agrees with the one given above, is shown to be the best. The third method leads to a different and less accurate formula.

Nothing in the investigation shows whether \( \Sigma x^2 \) is the sum of the squares of the true or of the computed errors. By later writers it has been generally taken as referring to the former. See 1816 Bessel, 1819 Young, 1823 Gauss, 1856 Peters, 1866 Bösch.


1818 Bessel. 'Fundamenta astronomiae pro anno MDCCLV deducta ex observationibus viri incomparablis JAMES BRADLEY in specula astronomiae Grenovicensi per annos 1750–1762 institutis.' Regiomonti, folio, pp. 325.

In pages 18–21 results of the computations of the mean and probable errors of the declination and right ascension of certain stars as deduced from the observations are given. Three sets of measurements, two of 300 and one of 470, are investigated as a test of the exponential law \( \varphi(x) = \frac{h}{\sqrt{\pi}} e^{-x^2} \), the theoretical number of errors between given limits being computed from Kramp's tables and compared with the actual number of residuals. A close agreement was found, and this may perhaps be called a practical proof of the principle of Least Squares.

Tables of logarithms of \( e^{-t} \int_t^\infty e^{-s^2} ds \) are given; *regula minimorum quadratorum* is several times applied; and in pages 118–123
methods are given for finding probable errors of quantities indirectly observed.

1818 Cauchy. Sur la méthode d'erreurs d'un grand nombre d'observations. [Paris], 4to.

1818 Laplace. 'Application du calcul des probabilités aux opérations géodésiques.' Connaiss. des Tems for 1820, pp. 422-440.


"Laplace shows how the knowledge obtained from measuring a base of verification may be used to correct the values of the elements of the triangles of a survey .... Laplace explains a method of treating observations which he calls the method of situation and which he considers may in some cases claim to be preferable to the most advantageous method explained in his fourth chapter."—Todhunter, History of Probability, pp. 611-612.


The theory of Least Squares and probable errors is used.


The elements and their probable errors are found by the help of Gauss's method of elimination.


A discussion of the length of the meridian between Perpignan and Formentera, two points distant about 460000 metres and joined by
26 triangles. Laplace shows that the probable error of the computed distance is 8.194 metres. The investigation is reproduced in the Third Supplement (1820), pp. 3–7.


1819 Paucker. ‘Über die Anwendung der Methode der kleinsten Quadratssumme auf physikalischen Beobachtungen.’ Mitau, 4to, pp. 32.

Published as a Gymnasium “Programm.” It contains an application of the Method of Least Squares to the determination of empirical formula for the expansion of fluids, the specific gravity of water, and the elasticity of steam. See 1825 Muncke.

1819 Walbeck. *Dissertatio de forma et magnitudine telluris, ex dimensis arcubus meridiani desintendiis.* Åbo, 8vo.

The first discussion of measurements of several arcs of meridians by the Method of Least Squares.

1819 Young. Remarks on the probabilities of error in physical observations, and on the density of the earth, considered, especially with regard to the reduction of experiments on the pendulum.’ *Phil. Trans. Lond.* for 1819, pp. 70–95.

Pages 70–83 are devoted to the Theory of Errors. Besides many interesting remarks it contains a method for finding probable errors, supposing that the probabilities of the several errors are proportional to the terms of the series \((1+1)^m\), \(m\) being an even number. His result for the probable error of the mean of \(n\) observations is

\[
R = 0.85 \frac{\sum x}{n\sqrt{n}}
\]

in which \(\sum x\) denotes “the mean of all the actual errors.” Young refers to Bessel, Gauss, &c., as having used only the sum of the squares of the errors in determining the probable error, and regards his method as more accurate. See 1816.

Contains discussions on the probable errors of the observations. See also *Abhandl. Akad. Berin* for 1825, pp. 23–35.


See 1819. At the end of the Supplement is an investigation of the general case of “observations assujetties à plusieurs sources d’erreurs.” See *Todhunter, Hist. of Probability*, p. 612.


In pages 3 and 4 Legendre says: “J’ai donné le premier deux méthodes sûres pour obtenir la solution à la fois la plus simple et la plus exacte, savoir: la méthode des corrections indéterminées . . . . ., et la méthode des moindres carrés qui paraissait alors pour la première fois.” See 1789 and 1805.

Pages 79–80 contain a “Note par M * * *” in which the honor of the discovery of the Method of Least Squares is claimed for Legendre on the ground of priority of publication, and in which Gauss although not mentioned by name receives several sharp hits.


This paper discusses at some length the different methods which may be imagined for finding a mean value, and concludes that the problem is indeterminate because it is impossible to render it independent of the law of facility of error, concerning which there may be “une infinité d’hypothèses.” It tries to determine a mean, first supposing that the probability of each given measurement is inversely proportional to the error committed and secondly supposing that that probability is inversely proportional to the square of the error, and concludes that the arithmetical mean can only be used when the observations differ but slightly among themselves.

The paper ends by offering a method for the correction of the arithmetical mean, which amounts to this: First find the average of the measured quantities and compute the residuals. Then take the reciprocal of each residual as the weight of its corresponding observation and find the mean of these weighted observations. Or as weights the reciprocals of the squares of the residuals may be taken. The new mean gives new residuals from which a second approximation may be made, and so on. In a note at the end, the editor (Gergonne) suggests that this approximation will always tend to one of the given measurements as the mean.

The Method of Least Squares is applied to the reduction of 149 observations from the transit of 1761 and 106 from that of 1769, and to the determination of probable errors. The most probable distance of the sun from the earth is found to be 20068800 German geographical miles with the probable error of 89150 miles.


The problem is: To determine the position of a point from horizontal angles taken at that point between other points whose position is exactly known. A numerical example is given in which the number of known points is five and the number of angles is six. This is often called Pothenot’s problem; see 1840 Gerling, 1866 Schott.


This memoir contains Gauss’s second Proof of the Method of Least Squares. The following quotation from pages 37–38 shows the hypothesis upon which the proof is based: “. . . integrale ∫xx q(x) dx ab x = − ∞ usque ad x = + ∞ extensum (seu valor medius quadrati x²) aptissimum videter ad incertitudinem observationum in genere definendam et dimetiendam, ita ut e duobus observationum systematisbus, quae quod errorum facilitatem inter se differunt, eae praeclime prestare censeantur, in quibus integrale ∫xx q(x) dx valorem minorem obtinet.” Gauss does indeed recognize and point out that this is only an arbitrary convention, but he justifies himself in adopting it on the ground that the definition of most advantageous results must be arbitrary, since the question is in its very nature indefinite, and that his definition leads to simple operations. The values of the unknown quantities found by his method he calls “valores maxime plausibiles.”

Gauss’s method leads to the rule of Least Squares, whatever be the number of observations or whatever be the law of facility of error provided only that positive and negative errors are equally probable. For analyses of his proof see 1844 Ellis and 1872 Glaisyer, the former regarding it as valid and the latter as unsatisfactory. In my opinion it is but little more than a begging of the question to assume that the mean of the squares of the errors is a measure of precision. See below 1825 Ivory, 1847 Galloway and 1872 Helmert.

The memoir contains an extended presentation of the practical features of the method and in this respect is of great value. The algo-
rithm for the solution of normal equations by the method of substitution (1816), the determination of weights and of formulas for mean error occupy the second part of the memoir. The value of the mean error of an observation of the weight unity being \( m = \sqrt{\frac{\sum x^2}{n}} \), Gauss takes \( \sum x^2 \) as referring to the true errors, and determines \( m = \sqrt{\frac{\sum \nu^2}{n - q}} \) as a practical formula, \( \sum \nu^2 \) referring to the computed residuals, \( n \) being the number of observations and \( q \) that of the unknown quantities; the investigation however is not very clear. See 1816 Bessel and Gauss, and 1856 Bienaymé.

For Gauss’s own account of the contents of these memoirs see the Göttingische gelehrte Anzeigen, Feb. 26, 1821 and Feb. 24, 1823. These reviews are reprinted in Vol. IV. of Gauss Werke, pp. 95–104. Gauss here states that in the year 1797 he found that the determination of the most probable values of observed quantities was impossible, unless the law of facility of error was known; and that since 1801 he had used the Method of Least Squares almost daily. See 1830 Riese.

1824 BERLIN. Explanatio methodi quadratorum minimorum. Lundæ, 4to.


The rule given is expressed by the formula

\[
m = 0.4769 \sqrt{\frac{\sum a^2 - \left(\frac{\sum a}{n}\right)^2}{\sqrt{2n}}}
\]

\( a_1, a_2, a_3, \ldots \) being the results of the \( n \) observations.


These memoirs are a commentary on Laplace’s fourth Chapter (1812) and seem to form a kind of translation which Poisson made of Laplace’s investigations for his own satisfaction. A large part of the memoirs are reproduced in his Recherches . . . , see 1837. See also 1830 Hauber, 1847 Galloway and Todd Hunter’s History of Probability, pp. 560–588. See Jahrb. Chem. u. Phys., Vol. IV, pp. 38–42.

‘Poisson confines himself to the case in which one element is to be determined from a large number of observations, but he treats this case in a more general manner than Laplace had done. Laplace had assumed that positive and negative errors were equally
likely, and that the law of facility of error is the same at every observation; but Poisson makes neither of these assumptions."—Tophunter in Trans. Camb. Phil. Soc., 1869, Vol. XI, p. 219.


A general statement of the method, which seems to have been elsewhere published in detail.


This paper contains two attempted proofs of the principle of Least Squares by methods independent of the Theory of Probability. The first, in page 5, rests on a vague analogy with the properties of a lever and is in the words of Ellis "little more than a petito principi concealed by a metaphor." The second, in pages 6-7, rests on the supposition that "the mean of the sum of the squares of the errors may be taken as a measure of the precision of the observations" which can scarcely be assumed as evident; this is similar to Gauss's proof of 1823.

Pages 81-86 are devoted to discussing the probability of errors. Ivory makes no distinction between true errors and residuals, and does not recognize that the probability of any definite error must be an infinitesimal. The remaining pages attempt to show that the Method of Least Squares cannot give the most advantageous or probable results unless the law of facility of error is \( \phi(x) = ce^{-x^2} \), and that Laplace's demonstration "whatever merit it may have in other respects is neither more or less general than the other solutions of the problem."

These two proofs are examined and exposed by Ellis in 1844, and the second proof with the criticisms on Laplace are analyzed by Glaisher in 1872. See also 1851 Hossard.


Contains matter from 1823 Gauss and 1819 Paucker.


The Method of Least Squares is used and formulae for finding mean errors of the results given.

These volumes contain investigations by Fourier on weights, probable errors, etc. See note by Demorgan in Lond. Assur. Mag., Vol. XIV, p. 89.


Contains Ivory's third attempted proof of the Method, which is still more absurd than those of 1825. See Ellis's analysis in his paper of 1844. See also below, 1830 Francoeur.


An application of a method, communicated to the author by Bessel, for the adjustment of geodetic triangulations by the use of Least Squares. See 1831 Hansen. See Abhandl. von Bessel (Leipzig, 1875), Vol. III, pp. 16-19.

1827 ———. Ueber die Theorie der Zuverlässigkeit der Beobachtungen und Versuche und der von derselben abhängigen Bestimmungen des Mittels aus gegebenen Zahlen. [Berlin].


Proposes the periodic function since so much used for discussing recurring phenomena, and illustrates its application to the determination of empirical formula. See 1864 Schott.


This memoir discusses a method for the combination of observations when the observed quantities are not expressed as explicit functions of the unknown quantities to be determined; and when the problem furnishes rigorous equations of condition which the determined values of the unknown quantities must exactly satisfy. The method of correlates for the adjustment of such conditioned observations is given and an algorithm presented for its use. A numerical example involving twenty-four observations subject to thirteen conditions illustrates the use of the formulæ. See the Göttinger gelehrte Anzeigen, Sept. 25, Trans. Conn. Acad., Vol. IV. 23 Oct., 1877.
1826, or Gauss Werke, Vol. IV, pp. 104–108 for Gauss’s own account of the contents of the memoir. See also 1830 Riese.

1828 Quetelet. Instructions populaires sur le calcul des probabilités, Bruxelles, 18mo.


Ivory’s proof of 1826 is given as a perfectly valid “démonstration.” This is repeated in the second edition.

1830 Hansen. Commentatio de gradus precisionis computatione. [Gotha, 4to.]

This was first printed “in einem Program, womit die hiesige Sternwarte [at Gotha] des Jubiläum des würdigen Olters gebeichtet hat.” It contained a method for finding the weights of values determined by the Method of Least Squares, which for a small number of unknown quantities is perhaps shorter than that of Gauss (1823). See Astron. Nachr., Vol. VIII, col. 462–463, and Encke in Berlin Astron. Jahrb. for 1835, p. 297.


to the Method of Least Squares.

The general term of the binomial theorem is shown to take the exponential form \( ce^{-n} \), when the number of terms is indefinitely great: See Laplace, *Théorie des . . . Prob.*, Chap. III, and 1837 Hagen. Formulæ for probability of errors between given limits and for probable errors are also developed. See 1836 Poisson.

1830 Riese. ‘[Eine Recension.]

A review of Gauss’s memoirs *Theoria combinationis . . . .*, 1823, and *Supplementum theoricae. . . .*, 1827, giving an account of their contents and a popular exposition of the subject.

1831 Cauchy. ‘Mémoire sur le système de valeurs qu’il faut attribuer à divers Élémens, déterminés par un grand nombre d’observations, pour que la plus grande de toutes les erreurs, abstraction faite du signe, devienne un minimum.’ *Jour. École Poly.*, Vol. XIII (cahier 20), pp. 175–221.

This was perhaps published about 1814, See Bull. Soc. Philom. Paris, for 1824, pp. 92–99. The methods of 1760 Bosovich and 1792 Laplace, are particular cases of Cauchy’s solution. See Laplace’s *Théorie . . . des Prob.*, Chap. IV, Art. 24.


A clear presentation of the Method of Least Squares according to Gauss, the proof being that of 1809.


A simplification of Gauss’s method of 1823 for finding weights. Not the same as 1830 Hansen. See 1832 Encke.

The measurements are discussed by Gauss's method of correlative (1828) whose algorithm is given in full, and also by a new method of Hansen. Rosenberger had in 1827 examined the same measurements.


A modification of the method of Laplace’s Third Supplement (1820), illustrated by a practical example.


After brief notices of five proofs of the Method of Least Squares, Encke gives the preference to Gauss’s of 1809. To establish this more rigidly he offers a demonstration to show that for direct observations the rule of the arithmetical mean gives the most probable result. This demonstration (in my opinion not a rigorous one) has been followed by many subsequent writers. It is repeated by Encke in the article quoted next below, and is particularly stated with confidence by Chauvenet in 1864. For criticisms see 1843 Reuschle and 1872 Glaischer. See also Encke’s later opinion, below under 1850.


These memoirs form a treatise on the Method of Least Squares, from which many text-books have been compiled.

The first memoir contains the proof of 1809 Gauss, reinforced by Encke’s attempted demonstration of the validity of the arithmetical mean, the discussion of weight and probable errors, and two tables of the probability integral $\frac{2}{\sqrt{\pi}} \int e^{-t^2} dt$, the first between the limits 0 and $t$, and the second between the limits 0 and 0.476986$r$ ($x$ being any error and $r$ the probable error). These were computed from Kramp’s tables of 1799 as quoted by Bessel in 1818. See Lond. Phil. Mag., 1871, Vol. XLII, p. 431, et sq. A translation of this first memoir and a reprint of the tables is given in Taylor’s Scientific Memoirs, 1841, Vol. II, pp. 317–389.
Encke takes \( y = \varphi(x) \) as the equation of the curve expressing the probability of error, and regarding \( x \) and \( y \) as continuous variables recognizes clearly that for a given error \( \varphi(x) \) must be an infinitesimal. But strange to say he deduces

\[
\varphi(x) = \frac{h}{\sqrt{\pi}} e^{-h^2 x^2}
\]

in which \( h \) is a finite quantity.

The second memoir contains the practical features of the method—Gauss’s algorithm for the solution of normal equations, Gauss’s (1823) and Hansen’s (1830 and 1831) methods of determining weights, etc. The third is devoted to the discussion of conditioned observations. At the time of publication these memoirs must have been of great value to students.


Adjustment of triangulations, determinations of probable errors, etc.


This valuable paper deduces rules for the adjustment of angles taken by the method of repetitions, and formulae for finding their weights and probable errors.

1834 Strootman. *Bevattelijk onderrigt in de Kansrekening, of de leer der waarschijnlijkheden.* Breda, 12mo.


When an empirical formula is to be derived from a great number of observations, Cauchy’s method may be used as easily, although perhaps with less accuracy than the Method of Least Squares. See below 1853 Bienaymé and Cauchy, 1842 Grunert and 1861 Schott. See an article by Bartlett in *Amer. Jour. Sci.,* 1862, Vol. XXXIV, pp. 27–33.

The general term of the binomial \((p+q)^m\), in which \(p+q=1\), is shown to approach the form \(\frac{e^{-x^2}}{\sqrt{2\pi mpq}}\), as \(m\) indefinitely increases. See 1830 Poisson.

1836 Rouvroy. Ueber die Methode der kleinsten Quadrate. Appendix to his Mechanik (Dresden and Leipzig, 8vo).


This work contains Hagen's proof of the Method of Least Squares. It is based upon the following hypothesis: "Der Beobachtungsfehler ist die algebraische Summe einer unendlich grossen Anzahl elementärer Fehler, die alle gleichen Werth haben und eben so leicht positiv, wie negativ sein können." This postulated, the proof consists in finding the general term of the expansion of \((\frac{1}{2} + \frac{i}{2})^m\), \(m\) being indefinitely large. The law of facility of error takes the form

\[q(x) = (\pi m)^{-\frac{1}{2}} e^{-\frac{x^2}{m}}\]

from which the principle of Least Squares at once follows.

The algebraic work of Hagen's method had in a somewhat different form been given by Laplace in the Théorie . . . des Probabilités (1812), p. 301, and in the articles 1830 Poisson and 1836 Poisson. Hagen's method is more elementary and in connection with his original hypothesis forms, I think, one of the best proofs of the Method of Least Squares.

Hagen's proof is given in the writings 1849 Wittstein, 1850 Encke, 1852 Dienger, and in a modified form 1846 Quetelet, 1865 Tait, 1866 Natani and others. Also see Price's Integral Calculus (Oxford, 1865), pp. 376–379. A discussion between Kummell and Merriman concerning this proof is now (Oct., 1877) going on in the Jour. Franklin Institute; see Vol. CIV, pp. 173–187, 270–274, et sq.


The matter of Poisson's previous memoirs on the law of great numbers is reproduced in Chap. III, and of those on the probability of the mean in Chap. IV.

1838 Bessel and Baeyer. "Grodmessung in Ost-Preussen und ihre Verbindung mit preussische und russische Dreiecksketten." Berlin, 4to, pp. xiv, 452.


“Ich werde nämlich die Entstehungsart der Beobachtungsfehler *aus ihren Ursachen*, zum Grunde des Folgenden machen. Wenn man anfängs die Fehler einer gewissen Beobachtungsart also aus *einer*, auf gegebene Art wirkenden Ursache hervorgehend betrachtet, so wird dadurch ihre jedesmalige Größe $x$ eine gegebene Funktion eines Arguments $E$, welches in derselben Art willkürlich ist, wie das Fallen eines Würfels. Aus dem Ausdruck $x = fE$ kann aber der Ausdruck $q(x)$ abgeleitet werden, . . . .”

Bessel seems to use the word *Ursache* in the sense of a *source of error*. His first investigation is of a case arising in the measurement of angles, where the error $x$ is related to the *Ursache* $E$ by the law $x = a \sin E$, every value for $E$ between the limits $\pm 1/\pi$ being equally possible. The law of facility of error he finds $q(x) = \pi^{-1}(a^2 - x^2)^{-1/2}$, and the probable error is 2.568 times greater than by the Method of Least Squares. An example where $x = aE^2$, which he shows may actually arise, gives also disagreeing results.

In the second part of the investigation we read: “Ich werde nun die Wahrscheinlichkeit eines Fehlers untersuchen, welcher aus der Zusammenwirkung mehrerer, von einander unabhängiger Ursachen entsteht,” each error so arising being considered as equally likely to be positive or negative. The result of the investigation is that the law of facility of error approximates closely to the exponential form $q(x) = ce^{-h^2x^2}$, provided that “viele Ursachen zur Hervorbringen des Beobachtungsfehlers zusammenwirken,” and “dass unter den, aus den einzelnen Ursachen hervorgehenden mittleren Fehlern, keiner die übrigen beträchtlich übertrete,” and these conditions Bessel thinks, are present in most observations.

This memoir is very valuable as showing that the exponential law of facility is not to be regarded as an *a priori* rule, free from exception, and as throwing new light on the condition under which it exists. On the whole it may be considered as a new proof of that law and hence of the Method of Least Squares.


The rule of Laplace here meant is a method for finding the probability of the error of the mean. The opening pages contain some interesting historical remarks, but the investigation itself is very long and tedious and seems to be of little value.


This popular book devotes a chapter to methods of finding weights and estimating probabilities of mean results. It contains tables of the error functions.

"The object of this paper is the correction of an oversight made both by Laplace and M. Poisson in pages 279 and 299 of their respective works on the Theory of Probabilities."


The mean errors of the adjusted elevations are deduced.


On the determination of the values and weights of observed quantities subject to conditional equations. A particular case of Hansen's problem is noted below under 1841.


Formulae for solving normal equations containing three unknown quantities and determining at the same time the weights. They are regarded by Bessel as shortening considerably the numerical work required by previous methods. See below 1873 Seeliger.


Computations arising in the formation of normal equations can be shortened by the use of tables of squares of numbers.


1840 BAVL. *Theorica e practica del probabile.* Bergamo, second edition, 2 vols, 8vo.


The first part of this memoir I have not seen. The second discusses the determination of empirical formulae for periodic observations, the calculation of probable errors and the correction of a field survey.

1840 GERLING. *Pothenot'sche Aufgabe, in praktischer Beziehung dargestellt.* Marburg, 8vo, pp. 32.

Contains mainly the solution of 1822 GAUSS. See 1866 SCHOTT.


Forty-four observations subject to sixteen conditions are adjusted. The method here employed for the combination of measurements of arcs of meridians has furnished a model for many subsequent investigations.


“Die Lage zweier unbekannter Punkte durch Hülle der Lage zweier bekannten Punkte zu bestimmen, ohne jene von diesen aus zu beobachten.”

1841 HÜLSSE. *Über die Berechnung von Beobachtungen durch die Methode der kleinsten Quadratssumme.* Leipzig, 4to.


Near the end of this memoir are remarks concerning the application of determinants to the solution of equations and to the finding of weights in the Method of Least Squares.


1842 FRIES. *Versuch einer Kritik der Principien der Wahrscheinlichkeitsrechnung*. Braunschweig, 8vo.


See 1835 CAUCHY.

1842 Littrow. ‘Theorie der kleinsten Quadrate.’ *Gehler’s Physi
nung,’ pp. 1200–1251.

A clear elementary exposition of the Method and its theory ac

gording to 1832 ENCKE.

1842 LORATSCHEWSKY. ‘Probabilité des résultats moyens tirés

164–170.

Serial expressions are deduced for the probability of error in the

mean of a limited number of observations.

1842 MERZ. *De theoria probabilitatris adhibita in physicam*. Monachii.

1842 PUISSANT. ‘Traité de géodésie, ou exposition des méthodes

trigonométriques et astronomiques, applicables à la mesure de la terre,
et à la construction du canevas des cartes topographiques.’ Paris, 2


and III pl. —First edition, see 1805.

The results of many of Puissant’s previous memoirs are here re

corded, and the Method of Least Squares is often used. The work is

one of the most valuable treatises on Geodesy extant.

1842 RAMUS. ‘Sur une question de probabilité relative aux cor


XXIV, pp. 80–84.

A determination of the probability "que dans un grand nombre
d’observations la différence des résultats moyens des hauteurs baro

métriques observées et réduites soit comprise dans l’intervalle λ."

1842 RÖBER. Experiment. Art. in *Handwörterbuch Chemie u.

Phys*. (Berlin, 8vo.)


Simplifies the formation of normal equations, etc., when the observations are made on the simple sums or differences of the unknown quantities.

1843 **Berkhan.** *Über die Methode der kleinsten Quadrate.* Blankenburg, 8vo.

1843 **Cournot.** *Exposition de la théorie des chances et des probabilités.* Paris, 8vo, pp. viii, 448. —German trans., Braunschweig, 1849, 8vo.

Chapters on the theory of means and the adjustment of observations are given. At the end is a table of values of the error function. Cournot's methods are often quite awkward.

1843 **Gerling.** *Die Ausgleichungs-Rechnung der praktischen Geometrie, oder die Methode der kleinsten Quadrate mit ihren Anwendungen auf geodätische Aufgaben.* Hamburg and Gotha, 8vo, pp. xix, 409, and 4 pl.

This book is dedicated to Gauss. The principle of Least Squares is assumed as the basis of the methods of adjustment. It consists of four parts; the first treats of direct, the second of indirect, and the third of conditioned observations, while the fourth discusses the form and number of conditional equations which need to be considered in the adjustment of triangulations.

The book is fully illustrated with practical examples, and contains the best systematized development of the application of the method to the treatment of simple geodetic measurements that has yet appeared. Gerling issued later (1845, 1855, 1862) some papers supplementary to the work, one of which contains a long list of errata.


This is a critical examination of the deduction of the expression for the probability of an error between given limits and of other points in the "Metaphysik" of the Method. Encke's proof of the validity of the arithmetical mean (1832) is also examined and found to be imperfect.

A reply to Reuschle's criticisms. Encke tries to explain that in the two expressions
\[ \sum_0^n \varphi(x) = 1 \text{ and } \int_0^n \varphi(x) \, dx = 1 \]
the symbol 1 has different meanings. He also shows that Reuschle had failed to understand his proof of the arithmetical mean.


Donkin attempts to establish "une espèce de Statique métaphysique sur des preuves de la même force que celles qu'on emploie en déduisant, à priori, les lois de la Statique ordinaire." The word "force" is taken to mean "tout motif qui nous porte à altérer la valeur attribuée à une quantité," and to these "forces" the principles of centre of gravity of bodies, of virtual velocities, etc., are applied, and the usual rules for the adjustment of observations by means of normal equations, weights, mean errors, etc., are deduced. No law of facility of error enters into the discussion.

Donkin's reasoning does not always seem to me clear or rigorous.


In this paper it is attempted "to bring the different modes in which the subject has been presented into juxtaposition, as that the relations which they bear to one another may be clearly apprehended."

Ellis first takes up Gauss's proof of 1809. He considers that Gauss is not justified in assuming that the rule of the arithmetical mean gives the most probable values, and he shows that besides mere convenience no satisfactory reason can be assigned why it should be so regarded. His remarks on this point are extremely valuable and sound. See 1872 Glaisher.

Laplace's demonstration is taken up and presented in a different but greatly simplified form, extended to the case of any number of unknown elements. Gauss's second proof of 1823 is also analyzed and the conclusion arrived at that "nothing can be simpler or more satisfactory." Lastly Ivory's three proofs (1825–6) are discussed and their illogical character clearly exposed. The paper is one of the most valuable in the theoretical literature of the subject.


An abridged method for the solution of certain forms of normal equations.
to the Method of Least Squares.


On the probable errors of interpolations in logarithmic tables.

1845 FISCHER. Die Theorie der Beobachtungsfehler und ihre Ausgleichung durch die Methode der kleinsten Quadrate. Pt. I of his Lehrbuch der höheren Geodäsie; Darmstadt, 8vo.


Three supplements to Gerling's book (1843). The first and second treat of the determination of points by angle measurements and the third of the precision of chain measurements.

1846 BRAVAIS. 'Analyse mathématique sur les probabilités des erreurs de situation d'un point.' Mém. ... par divers savans ... Inst. France, Vol. IX, pp. 255–332.

Observations being made on the coördinates of a point the probability that the apparent and true places are separated by a given distance is investigated, as also is the "valeur de la crainte mathématique de l'erreur" which is shown to be represented by the surface of a certain ellipsoid. The method requires the observations to be numerous.

In determining the probability that a point in a plane is located on an elementary area dy dx, Bravais takes the probabilities of the x and y deflections as independent. See 1850 HERSCHEL.


A discussion of a triangulation including ten stations at which thirty-five angles are observed subject to nineteen conditional equations. The methods of adjustment, of determining weights, of forming the conditional equations and all the steps of the process are given at length with great clearness. See Month. Notices Astron. Soc., 1843, Vol. V, pp. 262–264 for a full account of contents of the memoir.

These letters are of so elementary a character that S. A. R. is informed of the meaning of the signs + and −. They contain however a valuable popular exposition of the Theory of means and of the laws of error. For review of the book see 1850 Herschel.

At the end of the book is an appendix containing many valuable "Notes." In pages 375-380 a table of the terms of the binomial \((\frac{1}{2} + \frac{1}{2})^n\) for eighty terms on each side of the middle term is given and the method of its computation explained: these have since been called Quetelet's numbers, see 1869 Galton. In pages 384-387 it is shown that the general term of the binomial \((\frac{1}{2} + \frac{1}{2})^n\) approaches the exponential form \(e^{-\lambda x}\) as \(n\) indefinitely increases: this is similar to Hagen's investigation of 1837.

In pages 412-424 are printed three suggestive letters from Bravais, in which not only doubts are expressed as to the a priori necessity of the exponential law of facility of error, but examples are given to show that it is not universally true a posteriori. Bravais' view is that every cause of partial error gives rise to a distinct curve of facility and that the combination of these approaches the exponential form as a limit, partly because of the necessary law that positive and negative errors are equally likely, and partly because the combination itself must tend toward the binomial form. He alludes to Hagen's proof as not sufficiently rigorous.


A great part of this work is translated and adapted from Laplace's Théorie . . . des Prob., 1812, enriched by comments. The Method of Least Squares is treated at considerable length according to Laplace's method. At the end are given valuable tables, those of 1799 Kramp and 1832 Encke, and also factorial tables.


Poisson's analysis (1824) of Laplace's method is given, and also Gauss's proof of 1823.


A suggestive article. Let \(x\) be the true value of a quantity for which observations give the values \(a, b, c, \ldots\), then

\[x = f(a, b, c, \ldots)\]

and also we must have

\[x - m = f(a-m, b-m, c-m, \ldots)\].

Applying Taylor's theorem, Matzka deduces for \(x\)

\[x = \frac{ha + ib + kc + \ldots}{h + i + k + \ldots}\]
in which, \( h, i, k, \ldots \) are positive quantities whose values cannot be determined. If the observations are of equal weight and infinite in number the rule of the arithmetical mean is shown to follow.


An excellent practical paper.


A valuable work for geodetic engineers.


Contains a very clear exposition of Hagen’s demonstration (1837) of the law of facility of error and an excellent elementary presentation of the practical features of the method.


See also 1853 Wolf. I regret that I have been unable to see these articles.

A statement of probable errors of measurements of angles in U. S. Coast Survey triangulations. At the end of the paper, which seems to be a brief abstract of the original, there are some remarks by Peirce, Gould and Henry, which probably were incorrectly reported.


The object of this paper is to establish greater confidence in the practice of taking the arithmetical mean and in the validity of the exponential law of facility of error. Six of the ten problems of Lagrange’s memoir of 1774 are translated and a few comments added. Hagen’s demonstration of 1837 is also given in full and spoken of in very favorable terms. Encke alludes to the use of the “Erfahrungs-satz des Prinzips des arithmetischen Mittels” in his memoir of 1832 and says, “so blieb doch immer eine willkürliche Annahme übrig.” At the end of the article is an attempt to explain why \[ \int_{-\infty}^{\infty} q(x) \, dx = 1, \]
when \( q(x) \) is the probability of the error \( x \).


The observations discussed are statistical facts.


This paper contains in a popular form another proof of the Method of Least Squares. Supposing a stone dropped with the intention that it shall hit a mark on a horizontal plane, the reasoning assumes that the deflections from rectangular axes through the mark are independent; and deduces the exponential form \( ce^{-h^2x^2} \) for the law of deviation or error. From this the Method of Least Squares at once follows. This proof was put into algebraic language by Ellis (see below) and the unwarrantable character of the assumption clearly pointed out. See above 1846 Bravais, and below 1857 Boole, 1867 Thompson and Tait, 1872 Schlömilch, and particularly 1872 Glamisher. See also 1808 Adrain, where this proof was first given.

Herschel's proof is discussed and regarded as unsatisfactory. Laplace's method is also explained and defended. The paper is very interesting and valuable. See Glaisher, *Mem. Astron. Soc. Lond.*, 1872, Vol. XXXIX, pp. 112–113; also below 1851 Donkin.


On the centre of gravity of a system of points. See also Vol. VII, p. 407 and 454.


I have not seen Vol. II of this work and find no direct reference to its publication: it was probably published about 1853. Vol. I is devoted to the practical features of the method and Vol. II to the theory.

Vol. I assumes the principle of Least Squares and develops the methods of adjustments and comparison by probable errors, weights, etc. Numerous well chosen practical examples are given worked out in detail. The work abounds in historical information, and is the most complete textbook on the subject which has come to my notice.


Is it justifiable to use the Theory of Errors in finding the probable errors of quantities which are partly results of observation and partly deductions of formulæ?


Weight is defined as "the reciprocal of the square of the probable error." The paper discusses nine practical cases.


Appears to be of little importance.


Part III of this paper offers some critical remarks on the Theory of Least Squares with particular reference to Ellis's paper of 1850. Herschel's proof, it is said, "should be treated with respect." The Method of Least Squares may be used, if for no other reason, because "it is a very good method," as shown by Gauss's proof of 1823.


Ivory's first proof (1825) is here rediscovered under a slightly different form.


The substance of this and the preceding article is given in the following.


Contains new methods of computation, tests of accuracy, etc., which appear to be of little value.


After some interesting critical remarks, Laplace's analysis (1812) is given considerably simplified. According to Bienaymé's investigation the formula for probable error ordinarily used are only correct for one unknown quantity. For two, three and four unknown quantities, he finds that the probable errors should be respectively 1.746, 2.281 and 2.716 times larger than those given by the usual formula. His expression for the probability that an error is included between given limits differs sensibly for several unknown quantities from the common probability integral, particularly for limits but little removed from $x=0$. See 1873 Wrede.


1852 Biver. Théorie des moindres carrés établie par l'analyse pure. Bruxelles, 8vo.

Probably similar to his memoir of 1853.

Hagen’s demonstration (1837) is followed. The article forms an almost complete elementary treatise on the Method of Least Squares.

In the Supplement Hagen’s proof is abandoned, as resting on a questionable hypothesis and Gauss’s first proof is given in its place. Dienger appreciates clearly the defects of Gauss’s method, for he requires the number of observations to be infinite in order that the value given by the arithmetical mean shall coincide with the true value of the measured quantity.


Contains valuable practical formulæ for the computer.


See 1863 Börsch.


A clear exposition and solution of the problem. An example from 1849 Baeyer is discussed.


The law is shown to agree with the exponential law of facility of error.


This Criterion, founded on a principle of the Theory of Probability, proposes a method for determining by successive approximation, whether or not a suspected observation may be rejected. Tables are needed for its application: for these see below 1855 Gould and 1864 Chauvenet.

It is a fatal objection to this criterion that its use involves a contradiction of reasoning. The arithmetical mean, for instance, can only be used when the observations are all of equal weight, and the rejection of an observation which deviates considerably from the
mean asserts that the weights of the several values are not equal. See below 1856 Airy and Winlock, 1858 Stone and particularly 1872 and 1873 Glaisher.

The criterion has been used to some extent in the U. S. Coast Survey office, but has elsewhere, I believe, found no acceptance.


An interesting investigation, illustrated by a discussion of meteorological observations.


The principle of the arithmetical mean is proved according to 1832 Encke. The term “risque de erreur” is given to the function \( A + B \Sigma x^2 + C \Sigma x^4 D + \Sigma x^6 + \ldots \) and it is shown that this becomes a minimum when \( \Sigma x^2 \) is a minimum, and this condition is regarded as furnishing “les valeurs les plus plausibles des inconnues.” Formule for weights and mean errors are also developed.


It is maintained that the method of interpolation (1835 Cauchy) can be used for determining several unknown quantities from a redundant number of equations, with results nearly as accurate as by the Method of Least Squares.


It is maintained that the two methods differ “complètement,” and that even a contradiction exists. Cauchy’s method, it is said, is only a modification of the ordinary process of elimination, which assures no especial degree of probability to the results and which requires in practice as many operations as the Method of Least Squares. See below Cauchy.
Further remarks by Biernaymé referring to this discussion are given in pages 68-89, 197, 206 of Vol. XXXVII of the Comptes Rendus.


Gives an extract from the memoir of 1835, and maintains that in many investigations the method of interpolation is preferable to that of Least Squares.


The new method is claimed to be often the shortest, and the Method of Least Squares is said to give most probable results only under certain conditions.


In the latter part of the article the "restricteurs" are applied to the theory of Least Squares, and it is concluded that that Method furnishes most probable results only when the law of facility of error is the same for all the errors, when no limits can be assigned to the magnitude of an error, and when the probability of an error $x$ is proportional to $e^{-h^2x^2}$.


The conclusions of the preceding article are confirmed.


Shows that the most probable values may sometimes differ from those found by the Method of Least Squares.


An answer and review of some of Cauchy's articles: also maintains that the mean of the sum of the squares of the errors is under all circumstances a measure of the precision of the observations.

The system of factors is often very different from that given by the Method of Least Squares.


An abstract only is given. The result seems to be that the mean is worthy of great confidence.


"Bedingungsgleichung" is not here used in its usual sense. The paper contains an investigation of the value to be taken for $x$ when $A - Bx = 0$, and many values of $A$ and $B$ are given by observation. A certain form for $x$ is shown to involve the principle of Least Squares.

1853 Liagre. *Calcul des probabilités et la théorie des erreurs, avec des applications aux sciences d'observation en general, et à la géodésie en particular.* Bruxelles, 8vo.

This is a standard work on the subject.


See also 1849 Wolf.

1853 ———. *Tafeln zur Berechnung der Wahrscheinlichkeit des Vorkommens von Beobachtungsfehler.* Berlin, lith. MS., 4to, pp. 11.


Points out six new geometrical properties of the probability curve, and shows how its equation may be derived from a certain mechanical idea.


The first method is shown to possess a slight advantage.


This volume renders quite accessible the Latin memoirs of GAUSS. It contains translations of pages 208-220 of the Theoria motus . . . . 1809, pages 20-26 of Disquisito de . . . . Palladii, 1811, and the whole of the memoirs of 1816, 1822, 1823, 1826 and 1827. The memoirs of 1823 and 1827 form the bulk of the book, the others being added at the end as "Notes."


Historical and critical remarks made on presenting a copy of the above book to the Paris Academy.


Contains three pages of corrections and errata to his book (1843).


The probable error is given as ±0.072*.


A valuable theoretical discussion illustrated with practical examples.


"The small errors which are beyond the limits of human perception, are not distributed according to the mode recognized by the Method of Least Squares, but either with the uniformity which is the ordinary characteristic of matters of chance, or more frequently in some arbitrary form dependant upon individual peculiarities . . . ."


This and many of Schott’s following papers are very valuable, but they are not usually clear except to those who already understand the subject.

1856 **Schott.** 'Probable error of observation derived from observations of horizontal angles at any single station, and depending on directions.' *Rep. Coast Sur. U. S.* for 1854, pp. 86*-95*.

A discussion of 350 measurements taken at eleven stations.

1856 **Airy.** 'Letter from . . . [Remarks on Peirce's Criterion.]


The Criterion is strongly opposed. " . . . the whole theory is defective in its foundations and illusory in its results." It must be said, however, that some of Airy’s objections are not supported by very good logic.


Airy’s objections are taken up in detail; some of them are shown to apply equally well to the Method of Least Squares.


Let \( \sum z \) be the sum of the residual errors all taken positive, and \( n \) the number of direct observations of equal weight. Then Peters’ result is, that \( r \), the probable error of a single observation is,

\[
r = 0.845347 \frac{\sum z}{\sqrt{n(n-1)}}.
\]

See on this formula 1869 Lüroth, and 1875 Helmert.

Several methods are given and illustrated by numerical examples. The paper is of great value to a computer.


The process of deducing empirical formulæ for declination from observations by the Method of Least Squares is explained and illustrated, as also that of finding the probable errors of the constants which enter into such formulæ and of the computed results. Formulæ are deduced for fourteen stations. See also *Rep. Coast Survey* for 1855, p. 306, and for 1859, p. 298.


The precision is regarded as inversely proportional to the length of the line. Tables are given showing results for different kinds of ground. See 1863 BÖRSCH.


Points for which the probabilities of error are equal have an ellipse as locus. The most probable ellipse is assumed to be given by the Method of Least Squares. See *Build. math. phys. Acad. St. Péters.*, Vol. VII, p. 145.


Contains formulæ for the easy determination of the relative weights of three unknown quantities, and also remarks concerning the geometric significance of weights and their connection with determinants.

1857 BABINET and HOUSEL. 'Calculs pratiques appliqués aux sciences d'observation.' Paris, 8vo, pp. xvi, 388.

Mostly devoted to the theory of numerical approximations and interpolation formulæ. Two pages are given to the Method of Least Squares.

1857 BAUR. ————. *Programm des Stuttgarter Polytechnikursus für 1857.*

An article on determination of weights, etc.


It is shown that it is unnecessary to adhere in numerical computations to the strict letter of the Method of Least Squares, and that its application “.........requires the use of such numbers only, in the arithmetical processes peculiar to it and characteristic of the Method, as may be designated by one of the numerals 0, 1, 2, ... , 9, or of the fractions $\frac{1}{2}, \frac{1}{3}, \ldots , \frac{1}{9}$, or by a product of one of these numbers by an integral power of 10.”


In the first part of this memoir the rule of the arithmetical mean is discussed. “The result of Boole’s investigation is that if $n$ observations $p_1, p_2, \ldots , p_n$ be made upon the same quantity, then the most probable value of that quantity is a certain linear function of $p_1, p_2, \ldots , p_n$; this Boole demonstrates by his Calculus of Logic, and the analysis is of so peculiar a character that .... I feel scarcely qualified to express a decided opinion on its merits. ....the [final] result takes the form of the arithmetic mean.”—Glaisher, *Mem. Astron. Soc. Lond.*, 1872, Vol. XXXIX, p. 124.

In the latter part of the paper Herschel’s demonstration is reproduced and defended against the arguments of Ellis; see 1850. See Glaisher’s paper, just quoted, pp. 115.

1857 **Dienger.** ‘Ausgleichung der Beobachtungsfehler nach der Methode der kleinsten Quadratsummen. Mit zahlreichen Anwendung, namentlich auf geodätische Messungen.’ Braunschweig, 8vo, pp. viii, 168.

An excellent elementary text-book. Gauss’s proof of 1809 is followed, with the improvement that the probability of a definite error is an infinitesimal. See 1852 Dienger. Among the practical questions treated is the theory of repetitions in angle measurements.


Donkin observes that if two observations of an unknown quantity give $x=a$ and $x=b$, then the most probable value of $x$ is $\frac{1}{2}(a+b)$, but that we cannot regard the arithmetical mean of more than two observations as most probable. Taking $x$ to represent the true value of the unknown quantity Donkin says: “....it appears a natural and
obvious assumption (though I do not pretend that it is not an assumption) that the probability that \( x \) is between \( x \) and \( x + dx \) must be expressible in the form \( \psi \left( x - \frac{a+b}{2} \right)^{dx} \). From this the exponential law of facility of error is deduced.


Petzval concludes that the Method of Least Squares is entirely inapplicable in Optics. He proposes "die Methode der numerisch gleichen Maxima und Minima," which consists in making the sum of the \( 2m \) powers a minimum, \( m \) being a variable which tends toward infinity as a limit. The development and application of this method is to constitute the First Part of Vol. III of his work on Optics. This method was mentioned by Laplace in the *Théorie... des Probabilités*, p. 345.


A practical application of the Method of Least Squares.


On the error ellipse.


The first exhaustive discussion of the adjustment of indirect observations subject to conditional equations. See Jordan, *Elemente der Vermessungskunde* (Stuttgart, 1877, 8vo), p. 6.

A machine for calculating the sums $a_1^2 + a_2^2 + a_3^2 + \ldots$, and $a_1b_1 + a_2b_2 + a_3b_3 + \ldots$


Discussion by the use of determinants, etc.

1858 **Clarke.** *Ordnance Trigonometrical Survey of Great Britain and Ireland. Account of the Observations and Calculations of the Principal Triangulations and of the Figure, Dimensions and mean Specific Gravity of the Earth as derived therefrom.* London, 4to, pp. xvii, 782, with an Atlas of 28 Plates.

The whole triangulation is adjusted by the Method of Least Squares. The method of correlatives is explained at length and illustrated for a case involving seventy-four observations subject to thirty-nine conditional equations. An inspection of this book will give students an idea of the stupendous calculations which men of science undertake and execute.

1858 **Didion.** *Calcul des probabilités appliqué au tir des projectiles.* Paris, 8vo.

For an exposition of this subject see Sonnet's *Dictionnaire des mathématiques appliquées*, (Paris, 1867), pp. 1103–1108.

1858 **Grunert.** 'Drei Grössen $x$, $y$, $z$. deren Summe die gegebene Grösse $s$ ist, sind durch Messung bestimmt worden, und man habe dadurch für diese drei Grössen respective die Werthe $a$, $b$, $c$ erhalten. Da diese Werthe mit Beobachtungsefehler behaftet sind, und ihre Summe also im Allgemeinen nicht genau $s$ ist, so soll man dieselben so verbessern, dass die verbesserten Werthe genau die Summe $s$ geben, und die Summe der Quadrate der Verbesserungen ein Minimum ist.' *Archiv. Math. u. Phys.*, Vol. XXXI, pp. 480–481.


Errors subject to the, law of facility $q(x) = \text{constant}$ are particularly discussed.
1858 Kohler. ‘Die Landeseermessung des Königreichs Württemberg.’ Stuttgart, 8vo, pp. xii, 428, and 3 pl.

In the Appendix particularly are applications of the Method of Least Squares.


An excellent little text-book, in which Gauss's first demonstration is followed, with the improvement that the probability of a single error is an infinitesimal.

1858 Schindler. Ueber Fehler bei der Berechnung eines ebenen Dreiecks. Prag, 4to.


Weights and probable errors are found. The notation used on p. 333 is unusual and uncouth.


Translated from the Russian by Bienaymé: treat of interpolation by the Method of Least Squares.


On the adjustment of a quadrilateral whose sides and diagonals are measured.


On methods of abridging the computations.


Referring to the preceding article of Vorländer.
1858 Vorländer. *Ausgleichung der Fehler polygonometrischer Messungen.* Leipzig, 8vo, pp. 55.

Besides the Method of Leas Squares shorter approximate processes are given.


Opposes Vorländer's method given in the preceding.


The most probable value of the measure of precision $h$ is found to be $\sqrt{\frac{n-1}{2\Sigma x^2}}$ and not $\sqrt{\frac{n}{2\Sigma x^2}}$. See 1866 Börsch.


Only Laplace's *Théorie analytique des Probabilités* was consulted in preparing this book, and as a consequence it is unreadable except by those already thoroughly acquainted with the subject.


On Tchébycheff's method; see 1858 and 1859.


Contains among other matter a detailed history of the discovery of the personal equation. See 1866 Radau.


This is a free translation of Cauchy’s article of 1835. The method is illustrated by an example.


An excellent practical paper. The mean error is found to be proportional to the square root of the length of the line.


Contains additions and corrections to his book (1843).


It is concluded that the precision of angle measurements is proportional to the square root of the number of single observations, or to the number of repetitions; and that the precision of linear measurements is inversely proportional to the square root of the length of the line. The articles of Vorländer (see 1856) are discussed as also is Hartner’s (1852) treatment of this subject.

1863 Freedén. ‘Die Praxis der Methode der kleinsten Quadrate für die Bedürfnisse der Anfänger bearbeitet.—Erster Theil: Elementare Darstellung der Methode nebst Sammlung vollständig berechneter physikalischer, meteorologischer, geodätischer und astronomischer Aufgaben, welche auf lineäre und transcendente Gleichungen führen.’ Braunschweig, 8vo, pp. viii, 114.

An excellent little book, although some of the examples are rather long for a beginner. The principle of Least Squares is assumed.

On the various methods of forming and solving normal equations, of determining weights, etc.


This is mainly an abridgment of Encke’s memoirs of 1832. Encke’s demonstration of the rule of the arithmetical mean is in particular set forth with confidence. The reasoning showing that if \( q(x) \) is the probability of the error \( x \), \( q(x)dx \) is the rigorous probability that an error falls between \( x \) and \( x + dx \) is very illogical. At the end are valuable tables, two of the probability integral, and others for using Peirce’s criterion which is given nearly in the words of its author; see 1852 and 1856. Chauvenet adds an approximate criterion for the rejection of one doubtful observation, which is derived “directly from the fundamental formula upon which the whole theory of the Method of Least Squares is based.”

1864 Christoffel. 'Bestimmung einer Oberfläche durch lokale Messungen.' [Berlin], 4to.


This is a very valuable contribution to the theory of the arithmetical mean. It is shown that the average “is not merely the mean value of all the given values: it is also the mean supposition of all possible suppositions as to the mode of obtaining that value,” but that “the average is the most probable result only so long as we know nothing of the law of facility of error.” See Glaisher, *Mem. Astron. Soc. Lond.*, 1872, Vol. XXXIX, p. 90.

DeMorgan suggests the name “critical error” instead of probable error. The entire paper, like all of DeMorgan’s writings, is very interesting and suggestive.


See above 1828 Bessel.


A very elementary sketch of the Method.

1865 Gooss. 'Begründung der Methode der kleinsten Quadrate.' Kreutznach, 8vo, pp. 32.

A doctor's thesis. Contains a deduction of the law \( y = ce^{-ht}x^2 \) from the axioms that the curve is symmetrical, that it has the axis of \( z \) for an asymptote, that the equation must be a simple one, etc. The discussion is not very satisfactory.


The principle of the investigation is that an error arising from any source may be compared to the deviation from the most probable result of the number of white or black balls obtained by a great number of drawings from a bag containing equal numbers of white and black balls. The idea and the algebraic work is nearly the same as Quetelet's investigation of 1846. See 1872 Glaisher.

1865 Todhunter. 'A History of the Mathematical Theory of Probability from the time of Pascal to that of Laplace.' Cambridge and London, 8vo, pp. xvi, 624.

This work is invaluable to all students of the Theory of Probability and I have to acknowledge my great indebtedness to it in preparing the early part of this list. None but those who have undertaken such historical researches can form an idea of the immense amount of labor which must have been done in preparing a work like this of Todhunter.

Todhunter's analyses of the memoirs of Lagrange and Laplace are full and clear, and his commentary on Laplace's proof of the Method of Least Squares greatly simplifies the tedious investigations of the Théorie analytique des Probabilités. An account of Gauss's proof of 1809 is not given.


\( \Sigma x^2 \) being the sum of the squares of the residual errors and \( n \) the number of direct observations, the mean error has been taken as
\[ \sqrt{\frac{\sum x^2}{n}} \text{ and } \sqrt{\frac{\sum x^2}{n-1}}. \] The paper compares the results given by these two formulæ and accords the preference to the first. See 1816 Bessel, 1816 Gauss, 1823 Gauss and 1860 Dedekind.


Gauss's proof of 1808 and Hagen's of 1837 are given, and free use is made of Wittstein's work of 1849. The familiar equation expressing the law of facility of error appears here under the strange notation \( \varphi(\gamma) = Ce^{-\alpha^2 \gamma^2} \).


This is a translation from the *Moniteur scientifique* for 1865. It gives a detailed history and discussion of the subject of personal equation. See 1861 Peters.

1866 Schott. 'The problem of determining a position by angles observed upon a number of given stations. Solution of Gauss, with example.' *Rep. Coast Survey U. S.* for 1864, pp. 116–119.

The method is taken from Gerling's book, see 1840.


A valuable practical paper containing determinations of probable errors of observations, and the discussion of a case of adjustment in solving thirty-five normal equations and fifty-eight equations of correlations.


An investigation of relations between mean values of powers of errors and expressions for probability of errors.


The adjustment of indirect observations subject to conditional equations is fully treated.

This is rewritten from the edition of 1837. The proof of the law of facility is given substantially the same as before. The distinction between true and computed errors is not however clearly drawn.

The book is an excellent one for students and engineers, the greater part being of a practical character.


In the opening pages the law of facility \( q(x) = h \pi^{-\frac{1}{4}} e^{-hx^2} \) is deduced by Gauss’s method of 1809, \( x \) being regarded as the residual or computed error. The remarks on page 797 concerning probable errors seem to be true and valuable. The book is particularly full in the treatment of conditioned observations, and is a valuable one for geodetic engineers. See 1868 and 1869 for supplements to the work.


A continuation of Schott’s articles of 1855 and 1866, devoted mainly to the discussion of the probable errors of the linear and angular measurements of the triangulation. A comparison is also given of the measured lengths of three base lines with the lengths as computed through the triangulation. The paper is a very valuable one.


Herschel’s proof (1850) is given and spoken of as “simple and apparently satisfactory.” See below 1872 Schlömilch.


On “Espérances mathématiques,” their arithmetical means, etc.


The principle of Least Squares is proved, assuming that the arithmetical mean gives the most probable result.

1868 Hansen. *Fortgesetzte geodätische Untersuchungen, bestehend in zehn Supplementen zur Abhandlung von der Methode der*


1868 Hencke. 'Ueber die Methode der kleinsten Quadrate.' Leipzig, 8vo.
A doctor's dissertation containing historical and critical information relating to Least Squares. I regret that I have been unable to see a copy of it.

Contains applications of the Method of Least Squares.

1868 Miller-Hauenh. 'Höhere Markscheidekunst. Praktisch-theoretische Anleitung beim Markscheiden die vermeidlichen Fehler zu umgehen, die unvermeidlichen aber in einfacher und streng wissenschaftlicher Weise zu verbessern.' Wien, 8vo, pp. xii, 291.
A valuable book for mining engineers. In the first or practical part processes and their applications are given, while the proofs follow in the second part. An attempt is made to show that the arithmetical mean is the most probable result by the theory of combinations, all true errors being taken as equal. The term absolute weight is introduced for $h^2$. Gauss's first proof of the Method of Least Squares is given.

Contains a new demonstration of the validity of the arithmetical mean. See 1875.

Peirce's and Chauvenet's criteria (see 1852 and 1864) are regarded as troublesome to use and as based on an erroneous principle, and a criterion is proposed, which embodies, in the opinion of the author, the true grounds on which the judgment rests when rejecting discordant observations or mistakes. See below 1873 Glaisher and Stone.

On Encke's demonstration of the arithmetical mean, and on Schiaparelli's article of 1868.

1869 Faa de Bruno. *Traité élémentaire du calcul des erreurs, avec des tables stéréotypées, ouvrage utile à ceux qui cultivent les sciences d'observation.* Paris, 8vo, pp. vii, 72, xlv.

The tables are the best part of this ouvrage, but in that giving the values of $\sin$ there is at least one dangerous error. This was one of the first books on the Method of Least Squares which I read, and I take this opportunity to warn young students against it. The text is full of typographical and other errors and the subject is presented neither clearly or fully. The list of literature at the end does not contain the names of Legendre, Ivory, Encke, Bessel, Ellis, or Herschel, gives only one work by Hansen, and does not mention Gauss's *Theoria motus*.... The book deserves a speedy oblivion.


The exponential law of error is used in dividing mankind into grades of intellect. Quetelet's numbers (1846) are employed for this purpose and are given in the appendix.


Contains a method for finding probable errors from the $\frac{1}{n}(n-1)$ differences between $n$ observations taken two by two. See below Andrä.


Objects to Jordan's method on the ground that the differences are not independent. See 1872 for continuation of this discussion.

If \( n \) be the number of observations and \( q \) that of the unknown quantities, the probable error of a single observation is found to be

\[
r = \frac{0.8453 \Sigma x}{\sqrt{n(n-q)}}.
\]

This is an extension of the formula given by Peters in 1856. See 1876 Helmert.


A discussion of interesting experiments.


On page 9 of the First Supplement (1815) or on page 539 of the national edition of the *Théorie...des Prob.*, Laplace gave, without demonstration, a certain formula. "The primary object of this communication is to demonstrate the result which as I have stated Laplace merely enunciated.... A secondary object of the communication is to develop Laplace's own process of investigating the method of Least Squares; some of the results which he obtained for the case of two elements are here demonstrated to hold for the case of any number of elements."


An elementary sketch of the subject according to Gauss and Encke.


The object of this paper is to determine the law of facility of error on the hypothesis that an error arises from the joint operation of a large number of small sources of error, positive and negative errors not being equally probable. The investigation is not very clear.
to the Method of Least Squares.


This is translated from the German. It treats of the general term of the binomial \((\frac{1}{2} + \frac{1}{2})^m\) when \(m\) is very large.


This valuable work of reference contains a brief sketch of the history of the Method of Least Squares, with a short development of its theory according to 1832 Encke.


Points out that the Method was independently discovered and published by Adrain in 1808, and reprints a portion of the original investigation. Interesting biographical notes relating to Adrain are also given.

1871 Franke. ‘Die Dreiecksnetze vierter Ordnung, als Grundlagen geodätischer Detail-Aufnahmen zu technischer oder staatswirtschaftlichen Zwecken.’ München, 8vo, pp. xii, 281.

Numerous examples of adjustment are given. The theory and practice of the subject are presented in different chapters. It is an excellent book.


An account of Kramp’s, Bessel’s, Encke’s and other tables of the values of the probability integral, with a new table of values from \(x=3.00\) to \(x=4.50\).


An elementary sketch of the Method of Least Squares.


On the solution of normal equations, determination of weights, etc.


Deserves a place here as an aid in the Method of Least Squares on account of its great convenience, being arranged like logarithms.


1871 Zachariae. ‘De mindeste Quadraters Methode.’ Nyborg, 8vo, pp. viii, 234.

This is an excellent text-book. See review in *Jahrb. Fortschr. Math.*, Vol. III, p. 95.


The matter of this paper is mostly included in the following.


This is perhaps the most valuable of all the theoretical memoirs on our list, presenting as it does clear critical analyses of the principal proofs of the law \( p(x) = ce^{-hx^2} \) and of the Method of Least Squares. It has been of great value to me in preparing this list.
Adrain's first proof is examined at length and its reasoning shown to be defective. Then are analysed in order: 1. Gauss's first proof, including Encke's, DeMorgan's and Ellis's remarks on the arithmetical mean; 2. Laplace's method, Poisson's and Ellis's simplifications and Ivory's criticisms; 3. Gauss's second demonstration; 4. Herschel's proof, with Ellis's and Boole's criticisms thereon; 5. Tait's and similar proofs; 6. Donkin's proof of 1857. By means of the index at the end of this list the reader may refer back to these papers, where I have often quoted Glaisher's remarks.

It is considered unproved that the arithmetical mean gives the most probable result. Gauss's second proof is regarded as resting upon an arbitrary assumption, which practically assumes the point to be proved. Laplace's method is considered as giving the only correct and philosophical analysis of the question, and this Glaisher shows leads directly to the exponential law of facility, provided that the sources of error are very great in number and that positive and negative errors are equally likely. "Tait's proof" is found insufficient. The proofs of 1837 Hagen, 1838 Bessel, 1844 Donkin and 1870 Crofton are not discussed.

Peirce's criterion for the rejection of doubtful observations is regarded as "destitute of scientific precision." ".... under no circumstances have we a right to say an observation has no weight, though it may be better to give it none than to give it as much as the best." The method of assigning weights in such cases is hinted at; see below 1873 Glaisher.


1872 Helmert. 'Die Ausgleichungsrechnung nach der Methode der kleinsten Quadrate mit Anwendungen auf die Geodäsie und die Theorie der Messinstrumente,' Leipzig, 8vo, pp. xi, 348.

The exponential law is regarded as a law proved by experience. The arithmetical mean is said to be the most plausible value. Both the first and second proofs of Gauss are given, and the second is regarded as better and more general.

While the theoretical part of the book is not satisfactory, the practical part renders it valuable for geodetic engineers. Conditioned observations in particular are well treated.


A certain statute relating to errors in coinage is discussed.


A method less accurate than Least Squares.

The mean errors of 17 angle measurements and 21 base line measurements are given, the latter for a line one kilometer in length. The greatest mean error of a base line measurement is 63.2 mm. and the least 0.12 mm., the first being measured in 1739 and the second in 1860. This is one of those papers in which the results of long continued research and labor are expressed in a few lines.


On the method of deducing probable errors from the \(\frac{1}{2}n(n-1)\) differences of \(n\) measurements, given by him in 1872. See next article.


A simplification of demonstrations of two methods.


1872 Rumpen. 'Über den Zusammenhang der von Gauss begründeten Methode der kleinsten Quadrate mit der algebraischen Theorie der quadratischen Formen.' Bonn, 1872, 8vo, pp. 40.

A doctor's thesis. The conditions for minimum squares, etc., discussed by help of determinants.


Herschel's proof is taken from 1869 Thompson and Tait, and pronounced "einfache und anschauliche."


Offers without demonstration a new formula for the mean error of a base line measured in several portions. See 1873 Helmert and Jordan.


Contains some formulae for weights taken from 1838 DeMorgan, with an application to determining the probability of error in the atomic weight of thallium.


For continuation see below under 1876.


A comparison of the common formula with the one given by Zachariae in 1872, showing that the latter is less accurate.


Also a criticism on Zachariae's formula.


Concerning the solution of normal equations.


1873 Laurent. 'Traité du calcul des probabilités.' Paris, 8vo. pp. xii, 268.
This is intended as an introduction to the study of Laplace’s Théorie... des Prob. At the end is the best list of literature on the Method of Least Squares which I have seen.

"Given at several epochs, observed values of a quantity which varies uniformly with the time, to find by Least Squares the most probable values of the two constants which fix its value at any time." The analogy of the question with one of equilibrium in mechanics is pointed out. The solution of a system of linear equations by Least Squares may be represented in a similar way.

Gives an account of some interesting experiments "made to study the distribution of errors in the observation of a phenomena not seen coming on, as in the case of a transit, but sudden as in the case of the emersion of a star from behind the moon." The results are given graphically and show a decided approximation to the exponential law of facility.
In the ten pages of introduction new ideas are offered concerning a notation, "suggested by the study of the logic of relations."

See above 1840 Bessel.

1873 Stone. 'On the most Probable Result which can be derived from a number of direct Determinations of Assumed Equal Values.' Month. Not. Astron. Soc. Lond., Vol. XXXIII, pp. 570–572.
Shows that the arithmetical mean is the most probable results for
\(n+1\) observations, provided it is the most probable result for \(n\) ob-
servations, and as it is undoubtedly such for \(n=2\), "... it can be
shown to be generally true."

1873 WRÉDE. ‘Nagra anmärkningar rörande minste kvadrat-

The probable error is said to be not always 0.6745 of the mean
error but depends upon the number of unknown quantities involved.
See 1852 BIENAYMÉ. Bessel’s investigation of 1838 is also discussed.

1873 GLAISHER. ‘On the Rejection of Discordant Observations.’

It is here clearly pointed out how inconsistent is the rejection of
discordant observations by a criterion founded on the supposition of
the validity of the arithmetical mean. The idea first advanced by
DE MORGAN (Encyc. Metrop., 1847) that the mean is only an approxi-
mate value to be used in weighting the observations from which a
new mean is to be deduced, and so on, is here developed to a certain
extent. See 1821 ——. The criterion given by STONE in 1868 is
examined and pronounced untrustworthy and wrong.

1873 STONE. ‘On the Rejection of Discordant Observations.’

A reply to the preceding in which GLAISHER’s arguments are ex-
amined at length and the validity of the criterion maintained.
GLAISHER’s method for weighting observations is also discussed and
regarded “as mathematically unsound.”

1874 GLAISHER. ‘Note on a paper by Mr. STONE, “On the Rejec-
tion of Discordant Observations.”’ Monthly Notices, Vol. XXXIV,
p. 251.

1874 STONE. ‘Note on a Discussion relating to the Rejection of

1874 CANTOR. ‘Historische Notizen über die Wahrscheinlichkeits-
rechnung.’ Halle, 8vo, pp. 8.

This is of no value. GAUSS alone is mentioned in connection with
Least Squares.

1874 FECHNER. ‘Ueber die Bestimmung des wahrscheinlichen
Fehlers eines Beobachtungsmittels durch die Summe der einfachen
This is an extract from a memoir published in *Abhandl. Sächs. Gesell*; see below 1875. Fechner deduces the formula

\[ r_0 = \frac{1.195502 \cdot \sum x}{\sqrt{2n - 0.8548}} \]

\( n \) being the number of observations and \( \sum x \) the sum of the residuals all taken positively. See 1876 Hélmer.


A critical discussion of Gauss’s, Bessel’s and Jacobi’s methods for solving equations and determining weights.—See *Jahrb. Fortschr. Math.*, Vol. VI, p. 145.


This is a comprehensive and valuable work. In the parts relating to the theory of observations the proofs of 1809 Gauss and 1837 Hagen are given, as also the investigations of 1852 Bienaymé.

1874 Seidel. 'Ueber die Berechnung der wahrscheinlichsten Werthe solcher Unbekannten zwischen welchen Bedingungs-Gleich-

The conditional equations are regarded as having infinite weights.
This way of consideration appears to lead to a new method of solu-
tion.

1874 Seidel. Ueber ein Verfahren, die Gleichungen, auf welche
die Methode der kleinsten Quadrate führt, so wie lineäre Gleichungen
überhaupt, durch successive Annährung aufzulösen. Abhandl. Akad.
München for 1874, pp. .....

1875 Airy (W.). 'On the Probable Errors of Levelling; with
Engrs. for 1875, pp. ... —Engineering News, Vol. IV, pp. 77–78,
A very valuable practical paper.

1875 Baeyer. 'Ueber Fehlerbestimmung und Ausgleichung eines
177–188.
Bessel's method for adjusting a triangulation is applied to a con-
ected system of levels.

1875 Bellati. Intorno ad un modo di semplificare in alcuni casi
l'applicazione del metodo dei minimi quadrati al calcolo delle costanti

1865 Bienaymé. 'Application d'un théorème nouveau du Calcul

If a series of observations be arranged in the order of the measure-
ments, there are certain maxima and minima whose probable number
and position are given by the theorem. On pp. 458–459, 491–492 of
this volume of the Comptes Rendus are remarks by Bertrand on the
theorem.

1875 Dienger. 'Die LAPLACE'sche Methode der Ausgleichung
von Beobachtungsfehler bei zahlreichen Beobachtungen.' Denkschr.

The method is extended to the case of several unknown quantities.

The exponential law of error is regarded as an empirical law established by experience.


See title 1874 Fechner.


Reference is here made to articles by Tulla, Jordan and others on a graphical method of adjustment, whose titles I regret not to be able to give. See *Monatsbl. Badisch. Geometervereins* for 1875.


If all the men of a tribe were arranged in a row according to their heights, the middle man would have the mean height.

The curve \( y = ce^{-ht^2} \) is called an "ogive" and it is regarded as more likely to be approximately true of a statistical series than any other that can be specified a priori.


The formula given in 1856 by Peters is discussed, and shown to be correct only for direct observations. A new formula for probable error is proposed. See 1869 Lüroth and 1876 Helmert.


A discussion of 1444 observations to deduce an empirical law of error. The result is that the exponential law represents closely the probabilities of error.


Gauss's method (1816) is considered incorrect.

A reply to Mees’s article above.


Herschel’s proof is given. The numerical examples concern the probability of striking a target.


A demonstration that the arithmetical mean of direct observations gives “le seul résultat plausible et conciliable avec les exigences pratiques de la question.” See 1848 Matzka, 1868 Schiaparelli and 1876 Stone.


1876 Baurernfeind. 'Methode der kleinsten Quadrate.' *Ele-
An elementary sketch of the method.

1876 Chambers (C.) and Chambers (F.). 'On the Mathematical
Expression of Observations of Complex Periodical Phenomena; and
on Planetary Influence on the Earth's Magnetism.' *Phil. Trans.

1876 DeForest. 'Interpolation and Adjustment of Series.' New
Haven, 8vo, pp. 52.
A supplement to his memoirs of 1873. Besides other valuable
matter, methods for finding probable errors of adjusted terms are
given.

1876 Ferrero. *Esposizione del metodo dei minimi quadrati.*
Firenze, 8vo, pp. 234.

1876 Hagen. 'Untersuchungen über die gleichförmige Bewegung
des Wassers.' Berlin, 8vo, pp. 104.
All known observations on the mean velocity of rivers are dis-
cussed by the Method of Least Squares, and the most probable law
and formula for mean velocity are deduced.

1876 Helmert. 'Die Genauigkeit der Formel von Peters zur
Berechnung des wahrscheinlichen Fehlers directer Beobachtungen
113–132.

Simplifications are given of Helmert's formula of 1875, and the
formulae of Fechner, Jordan and André (see 1869–1874) are
discussed.

1867 Helmert. Ueber die Wahrscheinlichkeit der Potenzsum-
men der Beobachtungsfehler und über einige damit im Zusam-
192–218.

1876 Kummell. 'New Investigation of the Law of Errors of
Hagen's proof of 1837 is given abbreviated and improved, and the
usual rules for normal equations and probable errors are deduced.
The probability to commit no error at all is regarded as an absolute

The usual formula is compared with a new formula and shown to give larger values.


"The main object of this paper is to give rules for good observing derived from this theory." Hints for abbreviating computations are added.

1876 Skinner. 'Principles of Approximate Computations.' New York, 12mo, pp. v, 98.

Presents simple rules for conducting computations involving approximate quantities, in such a manner as to require the fewest figures and to show at once the degree of accuracy of the result.


Points out that some of the assumptions of Schiaparelli’s proof of 1875 agree with those of his own proof of 1873. The article is in English.


Venn’s views are: First, almost any regular and symmetrical method of treating the errors of observation will tend to approximate indefinitely toward the truth as the number of observations is indefinitely multiplied, and this whatever be the law of facility; secondly, the Method of Least Squares is the best method (upon the reasonably probable supposition of the universality of the exponential law), that is, it approximates quicker to the truth as the number of observations is increased than any other method; but its superiority over other reasonable methods is small in comparison with their common superiority over single observations.

"Jahrbuch über die Fortschritte der Mathematik." Berlin, 8vo. One vol. of about 750 pages appears yearly in 3 parts.

This invaluable publication has been of great use to me in preparing the above list for the years 1868–75. *Vol. VIII* (not yet issued) embracing the literature for 1876, will undoubtedly contain the titles of some writings on the Method of Least Squares which are not given here.
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ERRATA.

Page 172, line 21 from foot, for Remarks read Remarks.
" 173, " 11 " top, " Jäi " Jäi.
" 179, " 8 " foot, " unhekannten read unbekannten.
ADDENDA.

1876 Rüdiger. Die Methode der kleinsten Quadrate abgeleitet aus der Wahrscheinlichkeitslehre, und ihre Anwendung auf naturwissenschaftlichen Messungen. Frankfurt an der Oder, 8vo, pp. 48.

A doctor's dissertation.

1876 ———. 'Zusammenstellung der Literatur der Gradmessungs-Arbeiten.' Berlin, 4to, pp. 32.

This is drawn up by commissioners of the states and countries belonging to the European International Geodetic Survey. It contains references to about 380 writings on Geodesy, 200 of which are German, 84 English, 50 Italian and 24 French. The English and Italian literature is well presented, the German and French is not. Coming from such a source, this list should have been a great deal better.

The work is received just as this sheet goes to press, and the following are the additions which it renders necessary to the preceding pages:

The work by Bäeyer recorded on page 208 is a lithographed manuscript issued in 1867 or 1868. Two other parts on Geodesy were also published.


A lithographed manuscript by Bäeyer, entitled Untersuchungen über die Ausgleichung nach Winkel- und Seitengleichungen, was published in 1871.